Annals of Physics 323 (2008) 2971–2986

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Large momentum part of a strongly correlated Fermi gas

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ARTICLE INFO

Article history:
Received 6 March 2008
Accepted 7 March 2008
Available online 28 March 2008

PACS:
03.75.Ss
05.30.Fk
71.10.Ca

Keywords:
Energy
Momentum distribution
Adiabatic
Pressure
BEC–BCS crossover
BCS–BEC crossover

ABSTRACT

It is well known that the momentum distribution of the two-component Fermi gas with large scattering length has a tail proportional to $1/k^4$ at large k. We show that the magnitude of this tail is equal to the adiabatic derivative of the energy with respect to the reciprocal of the scattering length, multiplied by a simple constant. This result holds at any temperature (as long as the effective interaction radius is negligible) and any large scattering length; it also applies to few-body cases. We then show some more connections between the $1/k^4$ tail and various physical quantities, including the pressure at thermal equilibrium and the rate of change of energy in a *dynamic* sweep of the inverse scattering length.

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1. The theorem

Consider an arbitrary number of fermions with mass m in two spin states $\uparrow$ and $\downarrow$, having an s-wave contact interaction with large scattering length a ($|a| \gg r_0$ [1]) between opposite spin states, and confined by any external potential $V_{\text{ext}}(\mathbf{r})$. To discuss their momentum distribution, it is convenient to use a box with volume Ω and impose a periodic boundary condition. For a large uniform gas, Ω is its actual volume; but for a gas confined in a trap, it is better to use (conceptually) a very large box which contains the region of the gas. Now consider any stationary state, that is, any energy level or any incoherent mixture of various energy levels (such as an equilibrium state at any temperature [2]). The momentum distribution $n_{\mathbf{k}\sigma}$ ($\leqslant 1$ because of Pauli exclusion), that is, the average number of fermions with wave vector $\mathbf{k}$ and spin σ, has in general a well-known tail of the form $n_{\mathbf{k}\sigma} \approx C/k^4$ [3] at large $\mathbf{k}$, where

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0003-4916/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.aop.2008.03.005

$$C \equiv \lim_{\mathbf{k}\to\infty} k^4 n_{\mathbf{k}\sigma},$$

is independent of σ [4]. In this equation, the limit is physically restricted to $k \ll 1/r_0$, where r_0 is the effective interaction radius [1].

This tail determines how many large momentum fermions (on average) we can find, if we measure the momentum distribution of the gas. This measurement is practically possible and has already been carried out to some extent [5,6] in some recent experiments on the BCS (Bardeen–Cooper–Schrieffer superfluid) to BEC (Bose–Einstein condensate) crossover behavior of such a Fermi gas [7]. The number of fermions with momenta larger than $\hbar K$ is, on average

$$N_{(k>K)} = \frac{\Omega C}{\pi^2 K},$$

plus higher order corrections which are negligible at large K. Here $K \gg 1/l_{\uparrow\downarrow}$, $K \gg 1/\lambda_{\mathrm{dB}}$, but $K \ll 1/r_0$; $l_{\uparrow\downarrow}$ is the characteristic distance between two fermions in the opposite spin states [8], and $\lambda_{\mathrm{dB}} \sim \hbar/\sqrt{mk_{\mathrm{B}}T_{\mathrm{temp}}}$ is the thermal de Broglie wave length. $\hbar$ is Planck's constant divided by 2π, k_{B} is Boltzmann's constant, and T_{temp} is temperature [2].

If we tune the reciprocal of the scattering length [9] adiabatically (very slowly), but keep the confinement potential [such as a trap or an optical lattice (for ultracold fermionic atoms)] fixed, the total energy E (the sum of the kinetic energy, the interaction energy, and the external trapping energy) of the gas will change accordingly, and we have a quantity $\mathrm{d}E/\mathrm{d}(1/a)$, which we call the adiabatic derivative of energy (with respect to the inverse scattering length $1/a$).

The above two quantities, the magnitude of the $1/k^4$ tail, and the adiabatic derivative of energy, can be independently measured in experiment.

We show in this paper that these two quantities have a *simple*, *exact* and *universal* relation:

$$\frac{\hbar^2 \Omega C}{4\pi m} = \frac{\mathrm{d}E}{\mathrm{d}(-1/a)}, \tag{1}$$

and we shall call it *adiabatic sweep theorem*.

This theorem holds whenever the effective interaction radius r_0 [1] is negligible compared to the other relevant length scales of the problem: the scattering length, the average interparticle spacing, the thermal de Broglie wave length, and the characteristic length scale over which the external potential is inhomogeneous. In this limit ($r_0 \to 0$), the interaction between two fermions in opposite spin states is s-wave contact interaction, characterized by the scattering length only, and there is no interaction between fermions in the same spin state because of Fermi statistics and centrifugal barrier.

The theorem applies to any stationary state, such as the ground state, any excited state, and any finite temperature state [2] in thermal equilibrium. Generally speaking, if $|\phi_i\rangle$ are a set of energy levels of the system, then any mixed state described by a density operator $\rho = \sum_i \rho_i |\phi_i\rangle\langle\phi_i|$ [10] satisfies this theorem. The numbers of fermions in the two spin states $N_\uparrow$ and $N_\downarrow$, the external potential (as long as it is fixed), and the scattering length are all arbitrary, provided that the conditions specified in the previous paragraph are satisfied.

2. The proof

Although the theorem can be proved with conventional means as well, we shall use the formalism developed in [4]. This formalism, as we shall see below, allows us to prove this universal theorem in a universal (and simple) way.

The basis of the formalism developed in [4] is a pair of linearly independent generalized functions $\Lambda(\mathbf{k})$ and $L(\mathbf{k})$, satisfying

$$\Lambda(\mathbf{k}) = 1 (|\mathbf{k}| < \infty), \quad \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\Lambda(\mathbf{k})}{k^2} = 0, \tag{2a}$$

$$L(\mathbf{k}) = 0 (|\mathbf{k}| < \infty), \quad \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{L(\mathbf{k})}{k^2} = 1, \tag{2b}$$

$$\Lambda(-\mathbf{k}) = \Lambda(\mathbf{k}), \quad L(-\mathbf{k}) = L(\mathbf{k}). \tag{2c}$$