Annals of Physics 323 (2008) 3059–3064

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

The exact solution and integrable properties to the variable-coefficient modified Korteweg–de Vries equation

Yi Zhang *, Jibin Li, Yi-Neng Lv

Department of Mathematics, Zhejiang Normal University, Guest Road 288, Jinhua 321004, PR China

ARTICLE INFO

Article history:
Received 25 March 2008
Accepted 21 April 2008
Available online 3 May 2008

Keywords:
Soliton
vc-mKdV equation
Bäcklund transformation
Lax pairs
Bidirectional wave interaction

ABSTRACT

In this paper, a variable-coefficient modified Korteweg–de Vries (vc-mKdV) equation is investigated. With the help of symbolic computation, the N-soliton solution is derived through the Hirota method. Then the bilinear Bäcklund transformations and Lax pairs are presented. At last, we show some interactions of solitary waves.

1. Introduction

It is known that to find exact solutions of the nonlinear evolution equations (NLEEs) is always one of the central themes in mathematics and physics. In the past few decades, there are noticeable progress in this field. And various methods have been developed, such as the inverse scattering transformation (IST) [1], the Bäcklund transformation (BT) [2], Darboux transformation (DT) [3,4], Hirota's bilinear method [5,6], Wronskian technique [5,7], and so on.

However, to our knowledge, most of aforementioned methods are related to the constant-coefficient models. Recently, there has been a growing interest in studying variable-coefficient NLEEs [8,9]. When the inhomogeneities of media and nonuniformity of boundaries are taken into account in various real physical situations, the variable-coefficient NLEEs often can provide more powerful and realistic models than their constant-coefficient counterparts in describing a large variety of real phenomena, as seen, e.g., in the coastal waters of oceans [10], superconductors [11], space and laboratory plasmas [12] and optical-fiber communications [13].

* Corresponding author. Fax: +86 579 82298188.
E-mail address: zy2836@163.com (Y. Zhang).

0003-4916/$ - see front matter
doi:10.1016/j.aop.2008.04.012

It is found that the variable-coefficient KdV (vc-KdV) equation, which is from Bose–Einstein condensates and fluid dynamics, can be used to describe the water waves propagating in a channel with an uneven bottom and deformed walls and the trapped quasi-one-dimensional Bose–Einstein condensates with repulsive atom–atom interactions. In Ref. [14], Tian and Gao put their focus on a variable coefficient higher-order nonlinear Schrödinger (vcHNLS) model which can be used to describe the femtosecond pulse propagation, applicable to, e.g., the design of ultrafast signal-routing and dispersion-managed fiber-transmission systems, and new transformation was proposed from vcHNLS model to its constant-coefficient counterpart. The mKdV-typed equation, on the other hand, has been discovered recently, e.g., to model the dust-ion-acoustic waves in such cosmic environments as those in the supernova shells and Saturn's F-ring. In this paper, we will consider the variable-coefficient modified Korteweg–de Vries (vc-mKdV) equation, which can be read as

$$u_t + g(t)u_{xxx} + f(t)u^2u_x + l(t)u + q(t)u_x = 0, \tag{1.1}$$

where $g(t), f(t), l(t)$ and $q(t)$ are all analytic functions.

Eq. (1.1) has attracted considerable attention in many different physical fields including ocean dynamics, fluid mechanics and plasma physics [15,16]. For example, when $l(t) = 0$, it can be used to investigate the dynamics hidden in the plasma sheath transition layer and inner sheath layer.

For a generalized variable-coefficient NLEEs, it is not completely integrable unless the variable coefficients satisfy some specific constraint conditions. It has been shown that in Refs. [17] the constraint conditions on the coefficient functions for some variable-coefficient NLEEs to be mapped to the completely integrable constant-coefficient counterparts are precisely the same as those for such equation to possess Painlevé properties. Eq. (1.1) can pass the Painlevé test when $f(t)$, $g(t)$, $l(t)$ satisfy

$$f(t) = \frac{6}{C_0^2}g(t)e^{2\int l(t)\,dt}, \tag{1.2}$$

where $C_0 \neq 0$ is an arbitrary constant. That is to say, under the condition (1.2), Eq. (1.1) becomes integrable.

The organization of the paper is as follows. In Section 2, under the Painlevé-integrable condition, we obtain the bilinear forms and N-soliton solution of Eq. (1.1). We present the bilinear BTs and the Lax pairs for the vc-mKdV equation in Section 3. In Section 4, we will show the interaction of multiple solitons. Finally, the conclusions and discussions will be given in Section 5.

2. Bilinear forms and *N*-soliton solution of vc-mKdV equation

In this section, we will apply the Hirota method to construct N-soliton solution for the vc-mKdV equation. Based on this method, we first transform the vc-mKdV equation into bilinear forms, then construct N-soliton solution for it.

Under the transformation

$$u = IC_0e^{-\int l(t)\,dt}\frac{\partial}{\partial x}\ln\frac{F^*}{F} \quad (I = \sqrt{-1}), \tag{2.1}$$

where F^* is the conjugation of F. Eq. (1.1) can be transformed into the following bilinear forms

$$D_x^2 F \cdot F^* = 0, \tag{2.2a}$$

$$[D_t + g(t)D_x^3 + q(t)D_x]F \cdot F^* = 0, \tag{2.2b}$$

where D_t and D_x are the Hirota bilinear operators introduced by

$$D_x^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n a(x,t)b(x',t')|_{x'=x,t'=t}.$$

Provide that F can be expanded as perturbation series

$$F(x,t) = 1 + \sum_{j=1}^{N} F_j(x,t)\varepsilon^j, \tag{2.3}$$