Annals of Physics 323 (2008) 3074–3090

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Self-similar factor approximants for evolution equations and boundary-value problems

E.P. Yukalova [a], V.I. Yukalov [b,*], S. Gluzman [c]

[a] Department of Computational Physics, Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna 141980, Russia

[b] Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Joliot Curie, Dubna 141980, Russia

[c] Generation 5 Mathematical Technologies Inc., Corporate Headquaters, 515 Consumers Road, Toronto, Ont., Canada M2J 4Z2

ARTICLE INFO

Article history:
Received 28 March 2008
Accepted 19 May 2008
Available online 14 June 2008

PACS:
02.30.Hq
02.30.Mv
02.60.Lj
03.75.Lm
05.45.Yv

Keywords:
Evolution equations
Boundary-value problems
Solitons
Bose–Einstein condensate
Vortices

ABSTRACT

The method of self-similar factor approximants is shown to be very convenient for solving different evolution equations and boundary-value problems typical of physical applications. The method is general and simple, being a straightforward two-step procedure. First, the solution to an equation is represented as an asymptotic series in powers of a variable. Second, the series are summed by means of the self-similar factor approximants. The obtained expressions provide highly accurate approximate solutions to the considered equations. In some cases, it is even possible to reconstruct *exact* solutions for the whole region of variables, starting from asymptotic series for small variables. This can become possible even when the solution is a transcendental function. The method is shown to be more simple and accurate than different variants of perturbation theory with respect to small parameters, being applicable even when these parameters are large. The generality and accuracy of the method are illustrated by a number of evolution equations as well as boundary value problems.

1. Introduction

Differential equations appear in numerous problems of physics, applied mathematics [1–3] and many other branches of natural as well as social sciences (see, e.g., [4–7]). In the majority of cases,

* Corresponding author. Fax: +7 496 216 5084.
E-mail address: yukalov@theor.jinr.ru (V.I. Yukalov).

0003-4916/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.aop.2008.05.009

these equations are nonlinear and cannot be solved exactly, allowing for exact solutions only in a few exceptional instances. Then, in order to obtain approximate analytical solutions, one resorts to perturbation theory in powers of some parameters assumed to be small [1–3]. The resulting expressions are usually rather cumbersome and are difficult to analyze. Also, they form asymptotic series that are useful only for very small expansion parameters.

The validity of perturbative series can be extended to the finite values of parameters by reorganizing them with the help of the optimized perturbation theory [8]. The basic idea of this theory is to include in the initial approximation a set of auxiliary parameters which are transformed, at each step of perturbation theory, into control functions governing the series convergence [8]. The optimized perturbation theory has been successfully applied to a great variety of problems, providing rather accurate approximations (see review article [9] and references therein). It has also been applied to solving differential equations [10–17]. However, the weak point of this approach to finding the solutions of differential equations is that the optimization procedure results in very complicated equations for control functions, which are to be solved numerically. It is practically always much easier to solve the given differential equations numerically than to deal with the cumbersome optimization equations for control functions, anyway requiring numerical solution. This is why this approach, though being very useful for many other problems [9], has not found wide practical use for solving differential equations.

Another method of constructing approximate solutions is based on the self-similar approximation theory [18–26]. Then the solutions to differential equations can be represented in the form of self-similar exponential approximants or self-similar root approximants [27–34]. These approximants represent well those functions whose behavior at large variables is known to be either exponential or power-law, respectively.

In the present paper, we advocate a novel approach to constructing approximate solutions of differential equations. This approach is based on the use of *self-similar factor approximants* [35–37]. The mathematical derivation of the latter also rests on the self-similar approximation theory [18–26], but the structure of these approximants is rather different from the exponential and root approximants [27–34]. The structure of the self-similar factor approximants reminds that of thermodynamic characteristics near critical points [38,39]. This is why it has been natural to apply, first, these approximants to the description of critical phenomena [35–37,40]. It was shown that these approximants allow us a straightforward and simple determination of critical points and critical indices, agreeing well with the results of the most complicated numerical techniques, whose description can be found in articles [41–46] and books [47–49].

The self-similar factor approximants make it possible to define an effective sum of divergent series. Initially, these approximants were introduced [35–37] for summing the partial series of even orders, while the summation of odd-order series was not defined. Recently, the method was completed by defining the factor approximants of odd orders [40,50]. Now we have in hands a general and uniquely prescribed procedure for constructing the self-similar factor approximants of arbitrary orders. We show below that this procedure can be employed for finding very accurate approximate solutions to differential equations. In some cases, the method gives exact solutions, if these exist. The approach is very general, being applicable to linear as well as to nonlinear equations, and to initial-value as well as to the boundary-value problems. The principal difference of the method from other perturbation theories [1–3] is that we, first, represent the sought solutions as an asymptotic series in powers of the equation variable, but not in powers of a parameter. And, second, we extrapolate the given series to the whole range of the variable by means of the self-similar factor approximants. The advantage of the method is its extreme simplicity combined with the high accuracy of the obtained solutions.

2. Self-similar factor approximants

Here we describe the general procedure of constructing self-similar factor approximants as solutions to differential equations. We keep in mind ordinary differential equations, though the procedure can be generalized to partial differential equations. It would be unreasonable to plunge from the very beginning to complicated matters, but the principal idea should, first, be illustrated by not too com-