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Analytical study of the nonlinear Schrödinger equation with an arbitrary linear time-dependent potential in quasi-one-dimensional Bose–Einstein condensates

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ABSTRACT

Under investigation in this paper is *a nonlinear Schrödinger equation with an arbitrary linear time-dependent potential*, which governs the soliton dynamics in quasi-one-dimensional Bose–Einstein condensates (quasi-1DBECs). With Painlevé analysis method performed to this model, its integrability is firstly examined. Then, the distinct treatments based on the truncated Painlevé expansion, respectively, give the bilinear form and the Painlevé–Bäcklund transformation with a family of new exact solutions. Furthermore, via the computerized symbolic computation, a direct method is employed to easily and directly derive the exact analytical dark- and bright-solitonic solutions. At last, of physical and experimental interests, these solutions are graphically discussed so as to better understand the soliton dynamics in quasi-1DBECs.

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1. Introduction

Since the experimental realization of Bose–Einstein condensates (BECs) in ultracold atomic gases, more and more attention has been paid to the theoretical and experimental investigation on the properties of Bose gases [1,2]. One particularly interesting aspect in this area is the exploration of nonlinear excitations of matter waves, such as solitons and vortices [3,4], which have been observed in BECs. On the other hand, the four-wave mixing has also been already realized in BECs [5]. The existence of solitonic solutions is one of the notable features of nonlinear wave equations [6–8].

In the case of three-dimensional atomic BECs, the "macroscopic" wave function of the condensate obeys the following Gross–Pitaevskii (GP) equation [9]:

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V_{\rm ext}+\lambda|\Psi|^2\right)\Psi, \tag{1}$$

where $\int d\vec{r}|\Psi|^2=N$ is the number of atoms in the condensate, the coupling constant λ is related to the scattering length a (represents the interactions between particles, $a>0$ for repulsive interaction; $a<0$ for attractive interaction), namely, $\lambda=4\pi\hbar^2a/m$ with m denoting the mass of the atom, while $V_{\rm ext}$ is the external confining potential in which the BECs are formed. Actually, besides internal interactions, the macroscopic behavior of BECs is highly sensitive to external conditions, primarily to the external trapping potential. The external potential has many forms, such as the optical lattice potential, elliptic function potential, harmonic oscillator potential and double well potential [10]. As is well known, the nonlinearity in the GP equation results from the interatomic interactions and the GP equation has both dark- and bright-solitonic solutions which correspond to the repulsive and attractive nature of the interatomic interactions, respectively. In BECs, the bright-soliton [11] is expected for the balance between the dispersion and the attractive mean-field energy; while the dark-soliton [12] is a macroscopic excitation of the condensate characterized by a local density minimum and a phase gradient of the wave function at the position of the minimum. For a very long cigar-shaped BEC, the validity of Eq. (1) requires that the correlation length is much larger than the mean interparticle separation along the axial direction. When the transverse dimensions of the condensate are on the order of its healing length and its longitudinal dimension is much longer than its transverse ones, the GP equation reduces to the quasi-one-dimensional (quasi-1D) GP equation [13], which is a useful model for quasi-1D BECs.

The nonlinear Schrödinger equations (NLSEs) are of current importance in many research fields [6]. In this paper, we consider the mean-field model of a quasi-1D BEC trapped in a linear time-dependent potential which is given by the following NLSE [14]:

$$i\hbar\frac{\partial\Phi(\xi,\tau)}{\partial\tau}=-\frac{\hbar^2}{2m}\frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2}+\xi F(\tau)\Phi(\xi,\tau)+\frac{4\pi\hbar^2a}{m}|\Phi(\xi,\tau)|^2\Phi(\xi,\tau), \tag{2}$$

where $\Phi(\xi,\tau)$ is the macroscopic wave function of the condensate, $F(\tau)$ denotes the arbitrary function of time τ. With a dimensionless transformation: $t=\omega_0\tau$, $x=\sqrt{m\omega_0/\hbar}\xi$ and $u=\sqrt{\hbar/Nm\omega_0}\Phi$, Eq. (2) becomes the following dimensionless one:

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x^2}+\kappa|u|^2u+xf(t)u=0, \tag{3}$$

where $\kappa=-4\pi aN$, $f(t)=-\frac{\sqrt{\hbar m\omega_0}}{\hbar m\omega_0^2}F(\frac{t}{\omega_0})$. The interaction between the particles in the condensate is repulsive for $\kappa<0$ and attractive for $\kappa>0$, which corresponds to the s-wave scattering length $a>0$ and $a<0$, respectively.

It should be noted that much work has been done about the NLSE with a time-dependent external potential, i.e., Eq. (3). The exact soliton solutions of Eq. (3) have been constructed through the inverse scattering method [15] and the one-soliton solution has been obtained by employing Husimi's transformation [16,17]. With Hirota method developed, the one- and two-soliton solutions have been derived effectively in Ref. [18]. Based on the F-expansion method, some Jacobian elliptic function solutions have been presented in Ref. [14]. In this paper, with symbolic computation [6–8], we shall mainly conduct some analytical study of Eq. (3) from different viewpoints.