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Two-dimensional Schrödinger Hamiltonians with effective mass in SUSY approach

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ABSTRACT

The general solution of SUSY intertwining relations of first order for two-dimensional Schrödinger operators with position-dependent (effective) mass is built in terms of four arbitrary functions. The procedure of separation of variables for the constructed potentials is demonstrated in general form. The generalization for intertwining of second order is also considered. The general solution for a particular form of intertwining operator is found, its properties—symmetry, irreducibility, and separation of variables—are investigated.

1. Introduction

The non-relativistic quantum models with position-dependent mass were used many years ago for the effective description of different models which are more complicated than the standard "academic" one-particle Schrödinger equation (as examples, see [1]). In this sense, position-dependent mass is frequently dubbed as effective mass (EM). During the last years this sort of models became again very fashionable in the literature, mainly due to the growing interest in such complex problems

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as theoretical descriptions of nanodevices, motion in curved spaces, and models with pseudo-Hermitian Hamiltonians (for illustration, see [2]).

Different approaches to this branch of modern Quantum Mechanics were explored, and supersymmetric (SUSY) ones are among the most promising. In the framework of SUSY method, one may try to find functional dependence of effective mass and potential such that the corresponding Schrödinger equation is solvable or quasi-exactly (partially) solvable. These solvable models may be considered both as a base for perturbation expansions and as a laboratory for the study of qualitative properties. The SUSY method itself in Quantum Mechanics includes many different variants [3] and the SUSY intertwining relations seem to be the most promising among them.

In most general form, SUSY intertwining relations between a pair of partner Hamiltonians $H_{1,2}$ are:

$$H_1Q^+ = Q^+H_2; \tag{1}$$
$$Q^-H_1 = H_2Q^-, \tag{2}$$

where the mutually Hermitian conjugate intertwining operators $Q^\pm$ are called supercharges. These relations lead to the isospectrality of Hamiltonians $H_{1,2}$ up to possible zero modes of supercharges. This means that (again, up to zero modes of $Q^\pm$) the energy spectra of H_1 and H_2 coincide. Their bound state eigenfunctions are related (up to normalization factors) by the supercharges:

$$H_i\Psi_n^{(i)}(\vec{x}) = E_n\Psi_n^{(i)}(\vec{x}); \quad i = 1,2; \quad n = 0, 1, 2, \ldots; \Psi_n^{(2)} = Q^-\Psi_n^{(1)}; \quad \Psi_n^{(1)} = Q^+\Psi_n^{(2)}. \tag{3}$$

If either Q^+ or Q^- have some zero modes, and they coincide with the wave functions either of H_2 or of H_1, these wave functions are annihilated according to (3) and have no analogous states in the spectra of the partner Hamiltonian. Thus, from the intertwining relations one can find a pair of almost isospectral Hamiltonians, which sometimes can be solved (partially or exactly). By solving for the Hamiltonian we mean finding analytically its energy spectrum and corresponding wave functions.

We stress that the intertwining relations approach described above is very general, providing connections between pairs of spectral problems H_1, H_2. It depends neither on specific nature of spectral problems of the operators H_1, H_2, nor on the specific form of intertwining operators $Q^\pm$ [4–7].

Each Hamiltonian H_i included in the intertwining relation (3) has at least one symmetry operator $R_1 = Q^+Q^-, R_2 = Q^-Q^+$, which commutes with it:

$$[H_i, R_i] = 0; \quad i = 1,2 \tag{4}$$

Sometimes, these symmetry operators R_i are expressed in terms of the Hamiltonian H_i itself, and thus give no new information about the system. But otherwise, they describe indeed the symmetry of the model [6,7].

A variety of different realizations of the intertwining relations method was built during the development of SUSY Quantum Mechanics. In the original one-dimensional SUSY Quantum Mechanics with supercharges given by first-order differential operators, not more than one normalizable zero mode of $Q^\pm$ may exist, and the spectra of H_1, H_2 either coincide or differ by one bound state. The "symmetry operators" R_i coincide with the Hamiltonians in this case due to their factorization property: $H_1 = Q^+Q^-; H_2 = Q^-Q^+$.

In multi-dimensional SUSY Quantum Mechanics developed in [8] the supercharges of first order in derivatives are used also, but a set of d related intertwining relations between two scalar and $(d-1)$ matrix Hamiltonians (d is dimensionality of space) must be considered simultaneously.[1] A more interesting realization of intertwining relations was constructed for one- and two-dimensional systems by using supercharges of second [4,6,7] (and higher [5]) orders in derivatives (Higher order or Nonlinear SUSY Quantum Mechanics). In particular, for these systems the number of zero modes of $Q^\pm$ may be more than one (their number and properties are under the control). In the case of $d = 2$, many integrable systems with nontrivial symmetry operators R_i were built [6]. Most of these systems are not amenable to separation of variables.

[1] Alternatively, intertwining of first order between multi-dimensional *scalar* Hamiltonians leads [9] to the systems amenable to separation of variables.