Annals of Physics 324 (2009) 1388–1407

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Entanglement witnesses and geometry of entanglement of two-qutrit states

Reinhold A. Bertlmann *, Philipp Krammer

Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

ARTICLE INFO

Article history:
Received 8 October 2008
Accepted 3 January 2009
Available online 5 February 2009

Keywords:
Entanglement
Entanglement witness
Qutrit
Bound entanglement
Hilbert–Schmidt measure

ABSTRACT

We construct entanglement witnesses with regard to the geometric structure of the Hilbert–Schmidt space and investigate the geometry of entanglement. In particular, for a two-parameter family of two-qutrit states that are part of the *magic simplex*, we calculate the Hilbert–Schmidt measure of entanglement. We present a method to detect bound entanglement which is illustrated for a three-parameter family of states. In this way, we discover new regions of bound entangled states. Furthermore, we outline how to use our method to distinguish entangled from separable states.

1. Introduction

Entanglement is one of the most striking features of quantum theory and is of capital importance for the whole field of quantum information theory (see, e.g., Refs. [1–3]). The determination whether a given quantum state is entangled or separable is still an open and challenging problem, in particular for higher dimensional systems.

For a two-qubit system, the geometric structure of the entangled and separable states in the Hilbert–Schmidt space is very well known. Due to the Peres–Horodecki criterion [4,5], we know necessary and sufficient conditions for separability. This case is, however, due to its high symmetry quite unique and even misleading for conclusions in higher dimensions.

In higher dimensions, the geometric structure of the states is much more complicated and new phenomena like bound entanglement occur [6–11]. Although we do not know necessary and sufficient conditions à la Peres–Horodecki we can construct an operator – entanglement witness – which provides via an inequality a criterion for the entanglement of the state [5,12–14].

* Corresponding author. Fax: +43 1 4799 9515.
E-mail addresses: reinhold.bertlmann@univie.ac.at (R.A. Bertlmann), philipp.krammer@univie.ac.at (P. Krammer).

0003-4916/$ - see front matter
doi:10.1016/j.aop.2009.01.008

In this paper, we use entanglement witnesses to explore the geometric structure of the two-qutrit states in Hilbert–Schmidt space, i.e. we geometrically quantify entanglement for special cases, and present a method to detect bound entanglement. Two qutrits are states on the 3×3 dimensional Hilbert space of bipartite quantum systems. In analogy to the familiar two-qubit case, which we discuss at the beginning, we introduce a two- and three-parameter family of two-qutrit states which are part of the *magic simplex* of states [11,15,16] and determine geometrical properties of the states: For the two-parameter family, we quantify the entanglement via the Hilbert–Schmidt measure, for the three-parameter family we discover bound entangled states in addition to known ones in the simplex [9,15]. Finally, we give a sketch of how to use our method to construct the shape of the separable states for the three-parameter family.

2. Weyl operator basis

As *standard matrix basis* we consider the standard matrices, the $d \times d$ matrices, that have only one entry 1 and the other entries 0 and form an orthonormal basis of the Hilbert–Schmidt space, which is the space of operators that act on the states of the Hilbert space $\mathcal{H}^d$ of dimension d. We write these matrices shortly as operators

$$|j\rangle\langle k| \quad \text{with } j,k = 1,\ldots,d, \tag{1}$$

where the matrix representation can be easily obtained in the standard vector basis $\{|i\rangle\}$. Any matrix can easily be decomposed into a linear combination of matrices (1).

The Weyl operator basis (WOB) of the Hilbert–Schmidt space of dimension d consists of the following d^2 operators (see Ref. [17])

$$U_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}kn} |k\rangle\langle (k+m) \operatorname{mod} d|, \quad n,m = 0,1,\ldots,d-1. \tag{2}$$

These operators have been frequently used in the literature (see e.g. Refs. [11,15,18,19]), in particular, to create a basis of d^2 maximally entangled qudit states [18,20,21].

Example. In the case of qutrits, i.e. of a three-dimensional Hilbert space, the Weyl operators (2) have the following matrix form:

$$\begin{aligned}
U_{00} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
U_{10} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}, \quad U_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i/3} \\ e^{-2\pi i/3} & 0 & 0 \end{pmatrix}, \quad U_{12} = \begin{pmatrix} 0 & 0 & 1 \\ e^{2\pi i/3} & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 \end{pmatrix}, \\
U_{20} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}, \quad U_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{-2\pi i/3} \\ e^{2\pi i/3} & 0 & 0 \end{pmatrix}, \quad U_{22} = \begin{pmatrix} 0 & 0 & 1 \\ e^{-2\pi i/3} & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \end{pmatrix}.
\end{aligned} \tag{3}$$

Using the WOB, we can decompose quite generally any density matrix in form of a vector, called *Bloch vector* [17]

$$\rho = \frac{1}{d}\mathbb{1} + \sum_{n,m=0}^{d-1} b_{nm} U_{nm} = \frac{1}{d}\mathbb{1} + \vec{b}\cdot\vec{U}, \tag{4}$$

with $n,m = 0,1,\ldots,d-1$ ($b_{00} = 0$). The components of the Bloch vector $\vec{b} = (\{b_{nm}\})$ are ordered and given by $b_{nm} = \operatorname{Tr} U_{nm}\rho$. In general, the components b_{nm} are complex since the Weyl operators are not Hermitian and the complex conjugates fulfil the relation $b^*_{nm} = e^{-\frac{2\pi i}{d}nm} b_{-n-m}$, which follows easily from definition (2) together with the hermiticity of ρ. Note that for $d \geqslant 3$ not any vector $\vec{b}$ of complex components is a Bloch vector, i.e. a quantum state (details can be found in Ref. [17]).