Annals of Physics 324 (2009) 1158–1167

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

The geometry of Schrödinger symmetry in non-relativistic CFT

C. Duval [a,1], M. Hassaïne [b], P.A. Horváthy [c,*]

[a] Centre de Physique Théorique, CNRS, Luminy, Case 907, F-13288 Marseille Cedex 9, France
[b] Instituto de Matematica y Física, Universidad de Talca, Casilla 747, Talca, Chile
[c] Laboratoire de Mathématiques et de Physique, Théorique, Université de Tours, Parc de Grandmont, F-37200 Tours, France

ARTICLE INFO

Article history:
Received 9 December 2008
Accepted 26 January 2009
Available online 5 February 2009

PACS:
11.25.Tq
03.75.Ss

Keywords:
Non-relativistic conformal symmetry
Bargmann framework
Kaluza–Klein type theory
Non-relativistic AdS/CFT

ABSTRACT

The non-relativistic conformal "Schrödinger" symmetry of some gravity backgrounds proposed recently in the AdS/CFT context, is explained in the "Bargmann framework". The formalism incorporates the Equivalence Principle. Newton–Hooke conformal symmetries, which are analogs of those of Schrödinger in the presence of a negative cosmological constant, are discussed in a similar way. Further examples include topologically massive gravity with negative cosmological constant and the Madelung hydrodynamical description.

1. Introduction

Non-relativistic conformal transformations have initially been discovered as those space-time transformations that permute the solutions of the free Schrödinger equation [1,2]. In $D+1$ dimensional non-relativistic space-time with position coordinates $\mathbf{y}$ and time y we have, in addition to the (one-parameter centrally extended) Galilean generators, also

* Corresponding author.
E-mail addresses: duval@cpt.univ-mrs.fr (C. Duval), hassaine@inst-mat.utalca.cl (M. Hassaïne), horvathy@lmpt.univ-tours.fr (P.A. Horváthy).
[1] UMR 6207 du CNRS associée aux Universités d'Aix-Marseille I et II et Université du Sud Toulon-Var; Laboratoire affilié à la, FRUMAM-FR2291.

doi:10.1016/j.aop.2009.01.006

$$
\begin{aligned}
(\mathbf{y}, t) &\to (\mathbf{y}^*, t^*) = (\alpha\mathbf{y}, \alpha^2 t) \quad \text{dilatation}, \\
(\mathbf{y}, t) &\to (\mathbf{y}^*, t^*) = \left(\frac{\mathbf{y}}{1-\kappa t}, \frac{t}{1-\kappa t}\right) \quad \text{expansions},
\end{aligned} \tag{1}
$$

referred to as "non-relativistic conformal transformations". Added to the Galilean symmetries provides us with the Schrödinger group. Dilatations, expansions and time translations span an $o(2,1) \approx sl(2,\mathbb{R})$ subalgebra.

These rather mysterious extra symmetries have been identified as the isomorphisms of the structure of non-relativistic space-time [3,4]. For "empty" space, one gets, in particular, the (one-parameter centrally extended) Schrödinger group. Conformal transformations act as symmetries also when an inverse-square potential is introduced [1,5]. In $D = 3$ a Dirac monopole can be included [6,7], and in $D = 2$ one can have instead a magnetic vortex [8]. Full Schrödinger symmetry is restored for a matter field interacting with a Chern-Simons gauge field [9,10]. It can also be present in hydrodynamics [11,12]. See also [13].

Recently, the AdS/CFT correspondence has been extended to non-relativistic field theory [14–17]. The key point is to use the metric

$$
\bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu = \frac{1}{r^2}\left[d\mathbf{x}^2 + dr^2 + 2\,dt\,ds - \frac{dt^2}{r^2}\right] = \frac{1}{r^2} g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{2}
$$

where $\mathbf{x}$ is an d-dimensional vector and r an additional coordinate. The metric (2) is a $d+3$ dimensional relativistic space-time, conformally related to the pp-wave defined by $g_{\mu\nu}$ on the same manifold. The interesting feature of the metric (2) is that its *isometries* are the *conformal* transformations of $d+1$ dimensional non-relativistic space-time, with coordinates $(\mathbf{x}, t)$.

Below explain the construction and properties of this metric, and illustrate it on some physical examples.

2. Siklos spacetimes

The metric (2) belongs to the class of Siklos spacetimes [18–20], interpreted as exact gravitational waves traveling along *AdS*,

$$
\bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu = \frac{1}{r^2}\left[d\mathbf{x}^2 + dr^2 + 2\,dt\,ds - F(\mathbf{x}, r, t) dt^2\right]. \tag{3}
$$

$\xi = \partial_s$ is a null Killing vector. The metric (3) can also be presented as $\bar{g}_{\mu\nu} = \bar{g}^{\mathrm{AdS}}_{\mu\nu} - r^2 F \xi_\mu \xi_\nu$, generalizing the familiar Kerr–Schild transformation. For $F = 0$, (3) reduces to the anti-de Sitter metric.

The Einstein tensor of (3) satisfies

$$
\begin{aligned}
&\overline{G}_{\mu\nu} + \Lambda \bar{g}_{\mu\nu} = \rho \xi_\mu \xi_\nu, \\
&\rho = \frac{r^4}{2}\left(\partial_r^2 F - \frac{d+1}{r}\partial_r F + \Delta_{\mathbf{x}} F\right),
\end{aligned} \tag{4}
$$

where $\Lambda = -(d+1)(d+2)/2$. Hence, these spacetimes are solutions of gravity with a negative cosmological constant, coupled to light-like fluid. The only non-vanishing component of the Einstein–de Sitter tensor is $\overline{G}_{tt} + \Lambda \bar{g}_{tt}$.

The RHS of (4) is traceless, since it is the energy–momentum tensor of some relativistic fluid made of massless particles. When F satisfies the Siklos equation [19]

$$
\partial_r^2 F - \frac{d+1}{r}\partial_r F + \Delta_{\mathbf{x}} F = 0, \tag{5}
$$

then $\rho = 0$ and (3) is the AdS_{d+3} metric.

The effect of a conformal redefinition of the metric, $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ has been studied by Brinkmann [21]. Applied to our case, we see that the Einstein equation of the pp-wave $g_{\mu\nu}$ in (1) goes over, for $\Omega = r^{-1}$, to that with negative cosmological constant, appropriate for $\bar{g}_{\mu\nu}$.