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Anyonic order parameters for discrete gauge theories on the lattice

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We present a new family of gauge invariant non-local order parameters Δ_α^A for (non-abelian) discrete gauge theories on a Euclidean lattice, which are in one-to-one correspondence with the excitation spectrum that follows from the representation theory of the quantum double $D(H)$ of the finite group H. These combine magnetic flux-sector labeled by a conjugacy class with an electric representation of the centralizer subgroup that commutes with the flux. In particular, cases like the trivial class for magnetic flux, or the trivial irrep for electric charge, these order parameters reduce to the familiar Wilson and the ’t Hooft operators, respectively. It is pointed out that these novel operators are crucial for probing the phase structure of a class of discrete lattice models we define, using Monte Carlo simulations.

1. Discrete gauge theories and their excitations

In two-dimensional gauge theories, we distinguish two classes of particle-like excitations, which we call electric and magnetic. Electric excitations are labeled by non-trivial representations of the gauge group, and are either put into the theory as external charges or as dynamical fields in the action. The magnetic excitations are labeled by topological quantum numbers, on the classical level related to solitonic sectors of the gauge theory. In planar physics, these are the magnetic fluxes and non-abelian generalizations thereof. On the quantum level, these manifest themselves as particle-like excitations carrying these topological quantum numbers.

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The magnetic excitations may also carry electric charges, which leads to exotic particles called *dyons*. In the two-dimensional setting, these particle types may behave like *anyons* with fractional spin obeying highly non-trivial *braid statistics*. Generally speaking, quantum groups provide the language in which the quantum physics of two-dimensional anyonic systems is optimally casted and understood. In that sense, a complete classification of all sectors of a discrete gauge theory (DGT) can be achieved within the mathematical framework of the *quantum double* $D(H)$ [1] of the discrete gauge group H [2].

Experimental realizations of discrete gauge theories are still modest, though there are interesting proposals for implementing them in Josephson junction networks [3–5] and in systems of polar molecules in optical lattices [6]. Globally, symmetric implementations were explored in [7,8]. In a larger context, it should be pointed out that DGT's play a crucial role in Kitaev's [9] seminal paper on topological quantum computation, as they constitute the most elementary examples of Topological Quantum Field Theories. This simplicity follows from the fact that there is no underlying Conformal Field Theory involved as is the case for Chern-Simons theories.

1.1. Transformations on DGT states: quantum symmetry

We will not give a detailed account on the emergence of quantum group symmetry in DGT, this can be found in the literature [10], but do present a short summary of the basics to fix the notation and introduce some key concepts required later on.

Consider the following operators acting on states in the Hilbert space of a DGT. First, there is the *flux projection* operator, denoted by P_h, which acts as follows on a state $|\psi\rangle$:

$$P_h \mid \psi\rangle = \begin{cases} \mid \psi\rangle & \text{if the state } \mid \psi\rangle \text{ contains flux } h, \\ 0 & \text{otherwise}. \end{cases}$$

Second, we have the operator g, for each group element $g \in H$, which realizes a global gauge transformation by the element g:

$$g \mid \psi\rangle = |{}^g\psi\rangle,$$

where it should be noted that we have not yet modded out by the gauge group to obtain the physical Hilbert space.

These operators do not commute and realize the following algebra:

$$\begin{aligned} P_h P_{h'} &= \delta_{h,h'} P_h, \\ g P_h &= P_{ghg^{-1}} g. \end{aligned} \tag{1}$$

The set of combined flux projections and gauge transformations $\{P_h g\}_{h,g\in H}$ generates the quantum double $D(H)$, which is a particular type of algebra called a Hopf algebra.

In the following, we will see that its representations correspond one to one with the electric, magnetic and dyonic sectors of the discrete gauge theory.

1.2. Representation theory of the quantum double: particle spectrum

The representation theory of the quantum double $D(H)$ of a finite group H was first worked out in [11], but here we follow the discussion presented in [10] and follow the conventions of that paper.

Let A be a conjugacy class in H. We will label the elements within A as

$$\left\{{}^A h_1, {}^A h_2, \ldots, {}^A h_k\right\} \in A, \tag{2}$$

for a class A of order k. In general, the centralizers for the different group elements within a conjugacy class are different, but they are isomorphic to one another. Let ${}^A N \subset H$ be the centralizer for the first group element in the conjugacy class A, ${}^A h_1$.

The set ${}^A X$ relates the different group elements within a conjugacy class to the first:

$${}^A X = \left\{{}^A x_{h_1}, {}^A x_{h_2}, \ldots, {}^A x_{h_k} \mid {}^A h_i = {}^A x_{h_i}\, {}^A h_1\, {}^A x_{h_i}^{-1}\right\}. \tag{3}$$