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# A quantum exactly solvable non-linear oscillator with quasi-harmonic behaviour

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**Abstract**

The quantum version of a non-linear oscillator, previously analyzed at the classical level, is studied. This is a problem of quantization of a system with position-dependent mass of the form $m = (1 + \lambda x^2)^{-1}$ and with a $\lambda$-dependent non-polynomial rational potential. This $\lambda$-dependent system can be considered as a deformation of the harmonic oscillator in the sense that for $\lambda \to 0$ all the characteristics of the linear oscillator are recovered. First, the $\lambda$-dependent Schrödinger equation is exactly solved as a Sturm–Liouville problem, and the $\lambda$-dependent eigenenergies and eigenfunctions are obtained for both $\lambda > 0$ and $\lambda < 0$. The $\lambda$-dependent wave functions appear as related with a family of orthogonal polynomials that can be considered as $\lambda$-deformations of the standard Hermite polynomials. In the second part, the $\lambda$-dependent Schrödinger equation is solved by using the Schrödinger factorization method, the theory of intertwined Hamiltonians, and the property of shape invariance as an approach. Finally, the new family of orthogonal polynomials is studied. We prove the existence of a $\lambda$-dependent Rodrigues formula, a generating function and $\lambda$-dependent recursion relations between polynomials of different orders.


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## 1. Introduction

The non-linear differential equation

$$(1+\lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + \alpha^2 x = 0, \quad \lambda > 0 \tag{1}$$

was studied by Mathews and Lakshmanan in [1] (see also [2]) as an example of a non-linear oscillator (notice $\alpha^2$ was written just as $\alpha$ in the original paper); the most remarkable property is the existence of solutions of the form

$$x = A\ \sin(\omega t + \phi),$$

with the following additional restriction linking frequency and amplitude

$$\omega^2 = \frac{\alpha^2}{1+\lambda A^2}.$$

That is, the Eq. (1) represents a non-linear oscillator with periodic solutions that were qualified as having a "simple harmonic form." The authors also proved that (1) is obtainable from the Lagrangian

$$L = \frac{1}{2}\left(\frac{1}{1+\lambda x^2}\right)(\dot{x}^2 - \alpha^2 x^2) \tag{2}$$

which they considered as the one-dimensional analogue of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{1+\lambda\phi^2}\right)(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2),$$

appearing in some non-polynomial models of quantum field theory.

The non-linear Eq. (1) is therefore an interesting example of a system with non-linear quasi-harmonic oscillations. Recently, it has been proved [3] that this particular non-linear system can be generalized to the two-dimensional case, and even to the $n$-dimensional case and that these higher-dimensional systems are superintegrable with $2n-1$ quadratic constants of motion. Moreover, we point out that a geometric interpretation of the higher-dimensional systems was proposed in relation with the dynamics on spaces of constant curvature. It was also proved the existence of a related $\lambda$-dependent isotonic oscillator and that the two-dimensional oscillator, previously studied in [3], admits a superintegrable modification that corresponds to the $\lambda$-dependent version of the Smorodinski-Winternitz sytem [4,5]. In fact this means that the deformation introduced by the parameter $\lambda$ modifies the Hamilton–Jacobi equation but preserves the existence of a multiple separability.

On the other hand, Biswas et al. studied in 1973 [6] the ground state as well as some excited energy levels of the generalized anharmonic oscillator defined by the Hamiltonian $H_m = -\mathrm{d}^2/\mathrm{d}x^2 + x^2 + \lambda x^{2m}$, $m = 2, 3, \ldots$, and then they proposed the use of similar techniques for the analysis of the Schrödinger equation involving the potential $\lambda(x^2/(1+gx^2))$. Since then, this non-polynomial potential has been extensively studied by many authors [7–27] from different viewpoints and by making use of different approaches. In many cases the term $x^2/(1+gx^2)$ was introduced as a perturbation of an initial harmonic oscillator; that is, the potential to be solved was not $x^2/(1+gx^2)$ by itself but $x^2 + \lambda x^2/(1+gx^2)$ (exact analytical solutions have only been found for certain very particular values of the parameters $\lambda$ and $g$, see e.g., Refs. [12–24]). It is