Annals of Physics 369 (2016) 168–176

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

# Symmetric quadratic Hamiltonians with pseudo-Hermitian matrix representation

Francisco M. Fernández

*INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica, Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina*

## HIGHLIGHTS

- Symmetric quadratic operators are useful models for many physical applications.
- Any such operator exhibits a pseudo-Hermitian matrix representation.
- Its eigenvalues are the natural frequencies of the Hamiltonian operator.
- The eigenvalues may be real or complex and describe a phase transition.

## ARTICLE INFO

*Article history:*
Received 13 October 2015
Accepted 16 March 2016
Available online 31 March 2016

*Keywords:*
Symmetric operator
PT symmetry
Pseudo-Hermiticity
Quadratic Hamiltonian
Adjoint matrix

## ABSTRACT

We prove that any symmetric Hamiltonian that is a quadratic function of the coordinates and momenta has a pseudo-Hermitian adjoint or regular matrix representation. The eigenvalues of the latter matrix are the natural frequencies of the Hamiltonian operator. When all the eigenvalues of the matrix are real, then the spectrum of the symmetric Hamiltonian is real and the operator is Hermitian. As illustrative examples we choose the quadratic Hamiltonians that model a pair of coupled resonators with balanced gain and loss, the electromagnetic self-force on an oscillating charged particle and an active LRC circuit.



## 1. Introduction

In two recent papers Bender et al. [1] and Bender and Gianfreda[2] discussed two interesting physical problems: a pair of optical resonators with balanced gain and loss and the electromagnetic self-force on an oscillating charged particle, respectively. In both cases the authors resorted to

*E-mail address:* fernande@quimica.unlp.edu.ar.

http://dx.doi.org/10.1016/j.aop.2016.03.002
0003-4916/

Hamiltonians that are quadratic functions of the coordinates and momenta to describe the dynamics. They found that those quadratic Hamiltonians exhibit PT symmetry so that the quantum-mechanical counterparts show real spectra when PT symmetry is exact.

In the first case Bender et al. solved the Schrödinger equation in coordinate representation by writing each eigenfunction as the product of a Gaussian function times a polynomial function of the two coordinates and obtained suitable recurrence relations for the polynomials. In the second case Bender and Gianfreda [2] resorted to the approach proposed by Rossignoli and Kowalski [3] that consists in converting the quadratic Hamiltonian into a diagonal form by means of a canonical transformation of the creation and annihilation operators.

In two recent papers Fernández [4,5] proposed the application of a simple and straightforward algebraic method based on the construction of the adjoint or regular matrix representation of the Hamiltonian operator in a suitable basis set of operators [6,7]. The eigenvalues of such matrix representation are the natural frequencies of the Hamiltonian operator. Instead of invoking the PT symmetry of the problem the algebraic method takes advantage of the fact that those Hamiltonians are symmetric.

There are many other problems that can be modelled by quadratic Hamiltonians. For example, Schindler et al. [8] studied mutually coupled modes of a pair of active LRC circuits, one with amplification and another with an equivalent amount of attenuation, and found a remarkable agreement between theoretical results and experimental data. They argued that the gain and loss mechanism breaks Hermiticity while preserving PT symmetry. In a discussion of the bandwidth theorem Ramezani et al. [9] resorted to the same system of differential equations derived from Kirchhoff's laws.

The purpose of this paper is to apply the algebraic method to a general quadratic Hamiltonian in order to derive some general conclusion about its spectral properties. In Section 2 we outline the main ideas of the algebraic method. In Section 3 we apply the approach to a general quadratic Hamiltonian, derive the main result of this paper and illustrate the general results by means of two toy models. In Sections 4 and 5 we discuss the pair of resonators and the electromagnetic self-force mentioned above. In Section 6 we apply the algebraic method to the Hamiltonian associated to the differential equations for the active LRC circuit. Finally, in Section 7 we summarize the main results of the paper and draw conclusions.

## 2. The algebraic method

We begin the discussion of this section with some well known definitions that will facilitate the presentation of the algebraic method. Given a linear operator $A$ its adjoint $A^{\dagger}$ satisfies

$$\langle f|\, A^{\dagger}\, |f\rangle = \langle f|\, A\, |f\rangle^{*}, \tag{1}$$

for any vector $|f\rangle$ in the Hilbert space where it is defined. If $A^{\dagger} = A$ we say that the operator $A$ is symmetric. If $|\psi\rangle$ is an eigenvector of the symmetric operator $H$ with eigenvalue $E$

$$H\,|\psi\rangle = E\,|\psi\rangle, \tag{2}$$

then $\langle f|\, H\, |f\rangle = \langle f|\, H\, |f\rangle^{*}$ leads to $(E - E^{*})\,\langle\psi|\,\psi\rangle = 0$. Therefore, if $0 < \langle\psi|\,\psi\rangle < \infty$ then $E$ is real.

The algebraic method enables us to solve the eigenvalue equation for a symmetric operator $H$ when there exists a set of symmetric operators $S_N = \{O_1, O_2, \ldots, O_N\}$ that satisfy the commutation relations

$$[H, O_i] = \sum_{j=1}^{N} H_{ji} O_j. \tag{3}$$

Without loss of generality we assume that the operators in $S_N$ are linearly independent; that is to say, the only solution to

$$\sum_{j=1}^{N} d_j O_j = 0, \tag{4}$$

is $d_i = 0, i = 1, 2, \ldots, N$. It follows from Eq. (3) and $[H, O_i]^{\dagger} = -[H, O_i]$ that

$$H_{ij}^{*} = -H_{ij}; \tag{5}$$