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Lie algebraic approach to the time-dependent quantum general harmonic oscillator and the bi-dimensional charged particle in time-dependent electromagnetic fields

V.G. Ibarra-Sierra[a], J.C. Sandoval-Santana[a], J.L. Cardoso[b], A. Kunold[b,*]

[a] *Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa, Av. San Rafael Atlixco 186, Col. Vicentina, 09340 México D.F., Mexico*

[b] *Área de Física Teórica y Materia Condensada, Universidad Autónoma Metropolitana Azcapotzalco, Av. San Pablo 180, Col. Reynosa-Tamaulipas, Azcapotzalco, 02200 México D.F., Mexico*

HIGHLIGHTS

- We deal with the general quadratic Hamiltonian and a particle in electromagnetic fields.
- The evolution operator is worked out through the Lie algebraic approach.
- We also obtain the propagator and Heisenberg picture position and momentum operators.
- Analytical expressions for a rotating quadrupole field ion trap are presented.
- Exact solutions for magneto-transport in variable electromagnetic fields are shown.

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ABSTRACT

We discuss the one-dimensional, time-dependent general quadratic Hamiltonian and the bi-dimensional charged particle in time-dependent electromagnetic fields through the Lie algebraic approach. Such method consists in finding a set of generators that form a closed Lie algebra in terms of which it is possible to express a quantum Hamiltonian and therefore the evolution operator. The evolution operator is then the starting point to obtain the propagator as well as the explicit form of the Heisenberg picture position and momentum operators. First, the set of generators forming a closed Lie algebra is identified for the general quadratic

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[*] Corrupted author.

E-mail address: akb@correo.azc.uam.mx (A. Kunold).

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Microwave induced magnetoresistance oscillations

Hamiltonian. This algebra is later extended to study the Hamiltonian of a charged particle in electromagnetic fields exploiting the similarities between the terms of these two Hamiltonians. These results are applied to the solution of five different examples: the linear potential which is used to introduce the Lie algebraic method, a radio frequency ion trap, a Kanai–Caldirola-like forced harmonic oscillator, a charged particle in a time dependent magnetic field, and a charged particle in constant magnetic field and oscillating electric field. In particular we present exact analytical expressions that are fitting for the study of a rotating quadrupole field ion trap and magneto-transport in two-dimensional semiconductor heterostructures illuminated by microwave radiation. In these examples we show that this powerful method is suitable to treat quadratic Hamiltonians with time dependent coefficients quite efficiently yielding closed analytical expressions for the propagator and the Heisenberg picture position and momentum operators.

1. Introduction

The simple quantum oscillator is the building block of a very large number of well established physical models. Some of its most widespread applications are the atomic and molecular bonds that, under certain approximations, can be modelled by quadratic potentials. Time-dependent general harmonic oscillators (GHO), the most general version of a simple quantum harmonic oscillator, are not only relevant from the theoretical point of view but are also at the heart of many interesting applications as quantum optics [1–3], radio-frequency ion traps [4–11], quantum field theory [12], quantum dissipation (Kanai–Caldirola Hamiltonians) [13–19], and even cosmology [20,21]. Moreover, the GHO is the base to build more complex time dependent Hamiltonians such as the one of a charged particle subject to variable electromagnetic fields. This Hamiltonian has been applied to the study of interesting systems in fields such as quantum optics [22] and magneto-transport in lateral heterostructures subject to electromagnetic fields [23–26]. Since in many cases these Hamiltonians possess exact solutions, they have turned into key elements to understanding and modelling a wide variety of physical systems characterized by time dependent Hamiltonians.

The GHO Hamiltonian consists of a simple harmonic oscillator with time-varying coefficients, time-dependent linear terms on the position and momentum operator and an extra term proportional to the symmetrized product of the position and momentum operators. It can be described by following Hamiltonian

$$\hat{H} = \frac{1}{2}a\left(t\right)\hat{p}^2 + \frac{1}{2}b\left(t\right)\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) + \frac{1}{2}c\left(t\right)\hat{x}^2 + d\left(t\right)\hat{p} + e\left(t\right)\hat{x} + g\left(t\right), \tag{1}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators obeying the usual commutation relation $\left[\hat{x}, \hat{p}\right] = i\hbar$ and a, b, c, d, e and g are in general functions of time.

It has been studied by diverse mathematical methods such as the group-theoretical approach [27], the path integral approach [28], unitary transformations [7,29], and the Lewis and Riesenfeld [30] invariant theory [31–35,18]. The linear potential, a particular case of the GHO, has been treated through powerful methods as the Lewis and Riesenfeld [30] invariant theory [36–38], Feynman's path integrals [39–43], the generalization of the well known ladder operators [44], Laplace transform techniques [45], time-space transformation methods [46] and others [47,48]. The charge particle in electromagnetic fields has been studied through different methods that include the Lewis and Riesenfeld [30] invariant theory [49,50], path integral method [51], unitary transformation approach [52,19], and through quadratic invariants [53].