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Deformation quantization of the Pais–Uhlenbeck fourth order oscillator

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We analyze the quantization of the Pais–Uhlenbeck fourth order oscillator within the framework of deformation quantization. Our approach exploits the Noether symmetries of the system by proposing integrals of motion as the variables to obtain a solution to the $\star$-genvalue equation, namely the Wigner function. We also obtain, by means of a quantum canonical transformation the wave function associated to the Schrödinger equation of the system. We show that unitary evolution of the system is guaranteed by means of the quantum canonical transformation and via the properties of the constructed Wigner function, even in the so called equal frequency limit of the model, in agreement with recent results.

1. Introduction

Whenever one consider curvature terms, for example in general relativity or brane inspired models, one is faced naturally with field theories described by Lagrangians with higher order derivative terms. The Pais–Uhlenbeck fourth order linear oscillator, originally introduced in [1], is perhaps

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the simplest example and definitely the best known higher derivative mechanical system and, in particular, it has served as a toy model to understand several important issues related to Ostrogradsky instabilities emerging naturally in higher order field theories [2–8]. Recently, the Pais–Uhlenbeck oscillator has been used as a guide to study higher order structures associated to supersymmetric field theory [9], *PT*-symmetric Hamiltonian mechanics [10], and geometric models within the scalar field cosmology context [11]. In this sense, it is important to mention that naive quantization procedures for the Pais–Uhlenbeck model has to be enhanced in order to recover unitarity in a physical allowed sector. Our main motivation is thus to address for the Pais–Uhlenbeck oscillator, within the perspective of the deformation quantization formalism, the long standing problems associated to the non-unitarity of higher derivative theories. As we will see, within our formulation the unitarity is guaranteed straightforwardly, even in the equal frequency limit of the model, by the introduction of a well-defined Wigner distribution.

The framework of deformation quantization was introduced in [12] as an alternative approach to the problem of quantization. In this formalism one uses, as guidelines, the Dirac quantization rules in order to pass from classical physics to the quantum realm. As is to be expected, a consistency requirement for such a quantum theory is the existence of a classical limit, that is, a quantum system should reduce to its classical counterpart whenever the limit of $\hbar$, the Planck constant, tends to zero. From this perspective, the quantization of a classical system could be seen as a deformation of the algebraic structures involved in a parameter encoding the quantum nature associated to the system ($\hbar$ in our case). Furthermore, the quantization rules require that for any classical observable there is a corresponding quantum observable, and similarly, that the Poisson bracket corresponds to the quantum commutator. All these requirements can be achieved by replacing the usual product of the algebra of smooth functions on the classical phase space with an associative non-commutative product, depending on $\hbar$, such that the resulting commutator is a deformation of the Poisson bracket. We refer the reader to [13], where results on the explicit construction of maps between classical and quantum observables are explained in detail, to Refs. [14,15], where conditions on the existence of the star product are exposed, and to the reviews [16–18] for general aspects of deformation quantization, as well as for more recent developments.

Our approach is based on taking advantage of the symmetries inherent to the Pais–Uhlenbeck model in order to construct the Wigner function that contains the relevant quantum information of the system. In this manner, we show that there exists a couple of integrals of motion associated to Noether charges, which in turn serve as privileged variables in order to find the solutions to the $\star$-genvalue equation. Further, in order to obtain the quantum wave functions we consider both, classical and quantum canonical transformations. At a classical level we transform, in a standard way, the Pais–Uhlenbeck system to a simpler model composed of the difference of two uncoupled harmonic oscillators for which the Wigner function may be also obtained. We then use the latter Wigner function to obtain a wave equation which, by means of a quantum canonical transformation, may be used in order to obtain the quantum wave function for the original Pais–Uhlenbeck system. The resulting wave function is identical to the one obtained in [6] by a different reasoning. We also show that in the equal frequency limit of the model the source of the continuous spectrum can be traced out through a linear canonical transformation that maps the Pais–Uhlenbeck Hamiltonian to a Hamiltonian composed of a discrete spectrum part plus a continuous spectrum part, contrary to the unequal frequency case. Besides, we demonstrate that in the equal frequency limit the Wigner function is certainly unitary as consequence of composition of unitary transformations considered through the quantum canonical transformations. In this sense, our results explicitly manifest the ghost-free feature of the Pais–Uhlenbeck model, in complete agreement with [4,6].

The article is organized as follows. In Section 2, we include a brief review of deformation quantization in order to set our notation and to define some useful structures. We also consider quantum canonical transformations as they will be essential in our context to obtain the wave functions associated to the Pais–Uhlenbeck oscillator. In Section 3, we analyze the Wigner function for our model in terms of its integrals of motion and we identify the quantum wave equation. Also, in this section we detail the equal frequency limit for the Pais–Uhlenbeck oscillator. In Section 4, we include some concluding remarks. Finally, we address some technical issues related to the construction of the Pais–Uhlenbeck wave function in the Appendix.