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Null lifts and projective dynamics

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ABSTRACT

We describe natural Hamiltonian systems using projective geometry. The null lift procedure endows the tangent bundle with a projective structure where the null Hamiltonian is identified with a projective conic and induces a Weyl geometry. Projective transformations generate a set of known and new dualities between Hamiltonian systems, as for example the phenomenon of coupling-constant metamorphosis. We conclude outlining how this construction can be extended to the quantum case for Eisenhart–Duval lifts.

1. Introduction

Hamiltonian dynamics is a classic subject that, despite its age, does not cease to display new phenomena and provide insights. One relatively recent surprising one is that of coupling-constant metamorphosis [1], whereas under suitable conditions in a classical Hamiltonian system it is possible to promote a coupling constant to the role of a new dual Hamiltonian, while the original Hamiltonian becomes a coupling constant itself. This allowed the discovery of a series of dualities between systems previously considered different, as for example the Hénon–Heiles and Holt systems. A sizeable literature has arisen around the subject and we refer the interested reader to related articles.

Other types of dualities have been studied in the seemingly different context of the Eisenhart–Duval null lift of a natural Hamiltonian system: this is a higher dimensional description of a quadratic Hamiltonian system given in terms of null geodesics [2–6]. Such dualities have been employed to map the standard Kepler problem to a Dirac-type theory of gravity with time dependent gravitational constant [4], as well as to relate the motion of a particle in a electric and magnetic field to that into a new set of generally time-dependent fields [7–9], or to discuss transformations that are related to

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the appearance of a cosmological constant like term [10]. In all these cases, the dual systems are such that their Eisenhart–Duval metrics are related by a change of variables plus a conformal rescaling. The Eisenhart–Duval lift has found a renewed interest in recent years due to its application to the non-relativistic AdS/CFT duality, see for example [11–14] and the related vast bibliography. An important ingredient in the construction is the fact that the Eisenhart–Duval lift provides a Bargmann structure, which in turn defines a Newton–Cartan structure which has the interpretation of a non-relativistic spacetime on which it is possible to describe dynamics, including gravity. As an application the Schrödinger–Newton equation has been generalised in [15] using the Eisenhart–Duval lift to include a curved spatial background and non-inertial forces, and its Schrödinger–Newton maximal group of symmetries has been calculated. Both [10] and [15] discussed the role of the Schwarzian derivative in the conformal rescaling of the metric: the first noticed how it is related to a cosmological-constant type term, and the second in a related fashion required a condition of zero Schwarzian derivative of the (integral of the) conformal factor of the transformation in order to preserve the source term of the gravitational potential for the Schrödinger–Newton equation.

In this work we present a unifying point of view on the subjects above, in terms of what we call projective geometry of the dynamics. A quadratic Hamiltonian system admits null lifts which are not unique, of which the Eisenhart–Duval one is a specific example with special properties. To a null lift one can associate a section in a bundle of projective conics, which are defined as quadratic forms on a projective tangent space. The resulting dynamics is a theory of unparameterised geodesic curves, and the same section can be projected to seemingly different Hamiltonian systems, thus clarifying that the related dualities are just alternative descriptions of the same dynamics. The operation of changing the parameter on the geodesics induces a conformal rescaling of the lift metric, and since the underlying structure is independent of the choice of parameter a Weyl geometry is naturally induced. Möbius transformations play a special role since they are in a way the mildest possible kind of conformal rescalings, as they leave invariant the trace-free Ricci tensor. We describe in detail the dualities that act on an Eisenhart–Duval null lift, and provide a number of examples as well as a projective description of the coupling-constant metamorphosis and the Jacobi metric. We also describe the basic elements of the quantum version of the theory.

The rest of the work is organised as follows. In Section 2 we describe basic notions of the main objects we will deal with: null lifts, conformal rescalings and changes of parameterisation, Weyl geometry, the Schwarzian tensor. Section 3 contains the main theory: the projective tangent bundle, projective conics and Hamiltonian dualities. We also describe how conformal Killing tensors are natural objects in a Weyl structure. Section 4 deals with examples and applications of the general theory. We provide a new projective interpretation of the Jacobi metric and the coupling-constant metamorphosis, as well as a number of examples that show how previously known dualities fit into the description of the same projective object. Section 5 describes the quantum case applied to Eisenhart–Duval lifts, where projective dualities carry over from the classical to the quantum theory using the Yamabe operator. We finish in Section 6 with a summary and conclusions.

2. Preliminary notions

2.1. Natural Hamiltonians and null lifts

A natural Hamiltonian is by definition given by a quadratic function of the momenta

$$H = \frac{1}{2} h^{ij}(p_i + eA_i)(p_j + eA_j) + e^2 V, \tag{2.1}$$

where $\{q^i, p_i\}$, $i = 1, \ldots, n$ are conjugate variables, e a constant, $h^{ij}(q, t)$ an inverse metric, $V(q, t)$ a scalar potential, $A_i(q, t)$ a vector potential and the variable t is time. A null lift of this system corresponds to a Hamiltonian $\mathcal{H}$ that is: (i) non-degenerate, non-definite, quadratic and homogeneous in a new set of momenta that includes the original ones, (ii) such that the extra momenta are all conserved, and (iii) such that the original Hamiltonian can be recovered by setting $\mathcal{H} = 0$. An