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A Coulomb-related family of potentials having zero-energy bound states

Adam J. Makowski

Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5, 87-100 Torun, Poland

HIGHLIGHTS

- Central potentials are proposed.
- Zero-energy bound states are found.
- Point transformation is discussed.
- Quantum–classical comparison is made.

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ABSTRACT

We propose here a new large class of singular central potentials having zero-energy bound states for many values of angular momenta. The potentials are shown to be closely related to the standard attractive Coulomb interaction. Some of them admit the $E \neq 0$ bound states as well. A quantum–classical correspondence is also discussed.

1. Introduction

Bound and unbound quantum states with the total energy $E = 0$ were studied for both attractive as well as repulsive potentials. The studies, apart from being mathematically interesting, were also focused on applications of the states in many areas of physics. Some representative examples can be found, for instance, in the theory of cold-atom collisions [1], in the description of some modes in the Aharonov–Bohm solenoids [2], in the construction of some vortex lattices [3], in the discussion of charge carries in graphene quantum dots and rings [4], in the theory of Wannier–Mott excitons [5,6], in optics [7], and in a number of other fields of physics.

E-mail address: amak@fizyka.umk.pl.

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Exact solutions of the radial Schrödinger equation at the threshold energy $E = 0$ are met for almost all values of angular momentum only for restricted groups of potentials. The simplest case of this kind is given by the power-law potentials [8] $V_{DN} = -\gamma/\rho^{\nu}$ with $\gamma > 0$ and $\rho = r/r_0$. When $\nu < -2$ and $\nu > 2$ the solutions are square-integrable even though in the former case the classical trajectories are unbound. A similar group of potentials was considered in [9] in connection with the discussion of parabolic potential barrier in two dimensions.

Other interesting potentials, with many applications described e.g. in [10], are given by the formula [11–13] $V_{LDO} = -\omega\rho^{2k}/[2r_0^2\rho^2(1+\rho^{2k})^2]$. Now, the zero-energy solutions are square-integrable if the coupling constant $\omega > 0$ is 'quantized' according to some rule with k not necessarily an integer.

A similar condition is required for one more family of potentials $V_A = [\lambda^2/(2\mu + 1)^2][(\lambda^2/2)\rho^{\frac{1-2\mu}{\mu+1/2}} - C\rho^{\frac{-2\mu}{\mu+1/2}}]$ considered in [14,15], where λ and μ are parameters and, if C assumes some discrete values, the potentials have bound states at the threshold energy. They can be called the oscillator-related family of potentials since they are derivable by the point canonical transformation of the solvable harmonic oscillator problem at $E \neq 0$.

It is the purpose of the present paper to propose a large family of potentials having zero-energy bound states at $E = 0$ for many values of the angular momentum quantum number. To the best of our knowledge, the potentials [Eq. (6)] were not considered so far and, as we will show, they are closely related to the ordinary Coulomb problem. This will be shown with the help of a simple point transformation. Detailed calculations are given in Section 2 and the discussion of results in Section 3.

2. Quantum and classical solutions

In this section we shortly discuss the ordinary 2D Coulomb–Kepler model and, with its help, we shall find exact zero-energy bound states for a new class of singular central potentials. Finally, we shall derive corresponding to them classical orbits.

2.1. The 2D Coulomb–Kepler model

Let us remind first the radial Schrödinger equation for this case

$$\left[\frac{-\hbar^2}{2mr_0^2}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{l^2}{\rho^2}\right) - \frac{A}{\rho}\right]R(\rho) = ER(\rho), \tag{1}$$

where $A > 0$, $\rho = r/r_0 = \sqrt{(x/r_0)^2 + (y/r_0)^2} = \sqrt{X^2 + Y^2}$ and l stands for the angular momentum quantum number. This problem was discussed in a number of papers (see e.g. [16–18]) and its solutions, obeying the boundary conditions $R(\rho = 0) = R(\rho \to \infty) = 0$, together with eigenenergies, are respectively given as:

$$R_{nl}(\rho) = N_{nl}\exp(-\gamma_n\rho)\rho^{|l|}L^{2|l|}_{n-|l|-1}(2\gamma_n\rho), \tag{2}$$

$$E_n = \frac{-mr_0^2A^2}{2\hbar^2(n - 1/2)^2} = \frac{-\hbar^2\gamma_n^2}{2mr_0^2}, \tag{3}$$

where

$$\gamma_n = \frac{mr_0^2A}{\hbar^2(n - 1/2)}, \tag{4}$$

$$N_{nl} = \frac{(2\gamma_n)^{|l|+1}\hbar}{r_0^2}\sqrt{\frac{\gamma_n(n - |l| - 1)!}{2mA(n + |l| - 1)!}} = \frac{(2\gamma_n)^{|l|+1}}{r_0}\sqrt{\frac{(n - |l| - 1)!}{(n + |l| - 1)!(2n - 1)}}. \tag{5}$$

The symbol N_{nl} stands for normalization, n is the principal quantum number with the values $n = 1, 2, 3, \ldots$, and, for a given n, we have $|l| = 0, 1, 2, \ldots, n - 1$. In Eq. (2) L_s^{α} is the associate Laguerre polynomial. Relating m and e, respectively, to the mass and charge of electron, and assuming $Ar_0 =$