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Nonlinear power spectral densities for the harmonic oscillator

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ABSTRACT

In this paper, we discuss a general procedure by which nonlinear power spectral densities (PSDs) of the harmonic oscillator can be calculated in both the quantum and classical regimes. We begin with an introduction of the damped and the undamped classical harmonic oscillator, followed by an overview of the quantum mechanical description of this system. A brief review of both the classical and quantum autocorrelation functions (ACFs) and PSDs follows. We then introduce a general method by which the kth-order PSD for the harmonic oscillator can be calculated, where k is any positive integer. This formulation is verified by first reproducing the known results for the $k = 1$ case of the linear PSD. It is then extended to calculate the second-order PSD, useful in the field of quantum measurement, corresponding to the $k = 2$ case of the generalized method. In this process, damping is included into each of the quantum linear and quadratic PSDs, producing realistic models for the PSDs found in experiment. These quantum PSDs are shown to obey the correspondence principle by matching with what was calculated for their classical counterparts in the high temperature, high-Q limit. Finally, we demonstrate that our results can be reproduced using the fluctuation–dissipation theorem, providing an independent check of our resultant PSDs.

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1. Introduction

The harmonic oscillator, in which a particle is confined to a potential well that varies quadratically with position, has proven to be a very useful model in a number of classical and quantum systems. In the classical regime, the harmonic oscillator provides an excellent description of periodic systems such as a mass on a spring or a pendulum, as well as resonating electronic LC circuits. In the realm of quantum mechanics, an analogous model is successful in predicting the behavior of a number of bosonic systems, such as photons confined to an optical cavity or phonons in an elastic solid. In fact, the vacuum itself is thought to consist of an array of harmonic oscillators with a broad range of frequencies [1].

Often, a harmonic oscillator model is applied to a system in isolation, where we generally consider only linear effects. However, when we begin to consider coupling between harmonic oscillators, or with other systems altogether, nonlinearities begin to enter the model, leading to new physics. An example of this sort of interaction arises in cavity optomechanics, in which two harmonic oscillators, one describing an optical cavity and the other describing a mechanical resonator, are coupled to one another [2]. In this case, the motion of the mechanical resonator shifts the resonance frequency of the optical cavity, while the optics provide a radiation pressure force acting back on the mechanics. For moderate coupling, a simple linear model suffices, such that monitoring the electromagnetic field provides a readout of the linear motion of the oscillating mechanical device. However, as the interaction strength between the two systems increases, nonlinear coupling begins to occur, requiring that higher-order terms be added to the Hamiltonian [3–9]. This provides a method by which one can obtain direct access to higher-order powers of the mechanical resonator's motion. For instance, a number of experiments have demonstrated direct coupling to the square of the oscillator's displacement [3,4,9–11]. These types of measurements have generated significant interest, as they have been proposed as a method to perform quantum nondemolition (QND) measurements [12,13] of a mesoscopic quantum system [2,3,6,14–17], as well as other exotic two-phonon processes, such as mechanical cooling/squeezing [5] and optomechanically induced transparency [7,8].

In order to make such measurements effectively, a knowledge of the autocorrelation functions (ACFs) and power spectral densities (PSDs) corresponding to the nonlinear readout of the oscillator's motion is required. Though the first-order PSD is a well-known result [18–23], here we calculate a general PSD of any order for the quantum and classical harmonic oscillator, with a special focus on the linear and quadratic cases. The structure of this document is as follows. In Section 2, we provide a basic overview of the classical and quantum harmonic oscillators in the damped and undamped situations. Section 3 then provides a description of how to calculate the ACF and PSD of a classical time-dependent signal. Complementary definitions for a time-dependent quantum operator follow. Using the results of the previous two sections, Section 4 introduces a general procedure that can be used to calculate the classical and quantum PSDs of kth-order for the harmonic oscillator. Section 5 reviews the $k = 1$ case of the first-order PSD of the harmonic oscillator, which is immediately followed by an extension to the $k = 2$ case of the quadratic PSD in Section 6. Finally, we conclude the document by discussing how these PSDs can be used in the context of real experiments.

2. Background

2.1. Classical undamped harmonic oscillator

The model of the classical, undamped harmonic oscillator describes a system whose dynamics are governed by the following differential equation

$$\ddot{x} + \omega_0^2 x = 0, \tag{1}$$

where $x(t)$ is a time-dependent variable that in this case we choose to be the position of the oscillator and $\omega_0 = \sqrt{k/m}$ is the resonant angular frequency of the system, with k and m being the oscillator's spring constant and mass, respectively. The familiar oscillatory solution to this second-order differential equation is given by

$$x(t) = x_0 \cos(\omega_0 t + \phi), \tag{2}$$

where x_0 and ϕ are an arbitrary amplitude and phase of the motion set by the initial conditions.