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Solvable quantum lattices with nonlocal non-Hermitian endpoint interactions

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HIGHLIGHTS

- For a generic 1D Schroedinger equation a generalization of boundary conditions proposed.
- Non-Hermitian implementation studied via several discrete-coordinate quantum-lattice toy models.
- Bound states with real spectra found obtainable in closed form.
- Explicit physical Hilbert-space inner products constructed using two alternative methods.

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ABSTRACT

Discrete multiparametric 1D quantum well with $\mathcal{PT}$-symmetric long-range boundary conditions is proposed and studied. As a nonlocal descendant of the square well families endowed with Dirac (i.e., Hermitian) and with complex Robin (i.e., non-Hermitian but still local) boundary conditions, the model is shown characterized by the survival of solvability in combination with an enhanced spectral-design flexibility. The solvability incorporates also the feasibility of closed-form constructions of the physical Hilbert-space inner products rendering the time-evolution unitary.

1. Introduction

An active interest of researchers in quantum lattices *alias* quantum chain models *alias* tridiagonal-matrix Hamiltonians

$$H_{chain}=\begin{bmatrix} a_0 & b_0 & 0 & \dots & 0\\ c_1 & a_1 & b_1 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & c_{N-1} & a_{N-1} & b_{N-1}\\ 0 & \dots & 0 & c_N & a_N \end{bmatrix} \qquad (1)$$

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dates back to the pioneering applications of these Hamiltonians in organic chemistry in the thirties [1]. Still, their study belongs to the mainstream activities, say, in the tight-binding descriptions of electronic structures in solids [2], etc. The models of this type also helped to clarify some non-variational features of Green's functions in particle physics [3] and they are currently serving as an exemplification of several gain-and-loss-related phenomena in optics [4]. Last but not least, the use of models (1) threw new light on some questions of the emergence and confluence of Kato's exceptional points (KEP, [5]) in perturbation theory [6,7].

In our recent paper [8] we decided to pay a detailed attention to one of the simplest quantum models of this type, characterized by the next-to-trivial $(N+1)$ by $(N+1)$ matrix Hamiltonian

$$H^{(N+1)}(z) = \begin{bmatrix} 2-z & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2-z^* \end{bmatrix} \quad (2)$$

with a single complex non-real parameter $z \in \mathbb{C} \setminus \mathbb{R}$. We revealed that in spite of its manifest non-Hermiticity the model may be perceived as belonging to the conventional quantum theory in which it generates a unitary evolution of the system in question.

The results of paper [8] offered an explicit example and illustration of certain recent, not entirely conventional implementations of the abstract principles of quantum theory. In essence, these implementations are based on an innovative parity-times-time-reversal-symmetric ($\mathcal{PT}$-symmetric, PTS, [9]) presentation *alias* pseudo-Hermitian representation (PHR, [10]) *alias* three-Hilbert-space (THS, [11,12]) form of the formalism of quantum theory (see Appendix A for a compact summary of these ideas).

In paper [8] we just confirmed that in practice, the costs of the PTS/PHR/THS enhancement of the flexibility of models may be reasonable and acceptable. Our main result was that we managed to fulfil the technically most difficult task of the PTS/PHR/THS theory and reconstructed the nontrivial "standard" representation $\mathcal{H}^{(S)}$ of the physical Hilbert space in which the manifestly non-Hermitian Hamiltonian (2) becomes Hermitized so that, in other words, the evolution in time acquires the standard unitary interpretation.

A formal simplicity of the discrete model (2) may be perceived as inherited from its differential-operator predecessor of Ref. [13]. In parallel, it should be emphasized that these two models also share the physical motivation and an immediate phenomenological appeal which was verbalized in Ref. [14] and which appeared to lie in the existence of a close relationship between the bound-state and scattering experimental data (cf. also Refs. [15,16] in this context).

All these observations provoked, naturally, a search for the generalizations which would eventually go beyond the tridiagonal matrix structure (1) of the model. A few new results obtained in this direction will be presented in what follows. First of all, our inspiration by PTS matrix (2) will lead, in a way described in Section 2, to its partitioning and to its subsequent multiparametric PTS generalization. In Section 3 we shall illustrate some descriptive merits of spectra provided by this generalization. The survival of solvability of the model will finally be demonstrated, constructively, in Section 4.

Due to the non-Hermiticity of the Hamiltonian, we will have to address also the above-mentioned problem of construction of the correct physical representation Hilbert space. This will be done in several sections. Firstly, the general recipe will be presented in Section 5. Two alternative methods of construction of the correct inner-product metric Θ will be described. We shall point out that both parts of the construction of the bound state solutions (viz., the construction of the wave functions and of the metric) appear closely interconnected in a way which may be perceived as an explanation why, in spite of its enhanced flexibility, our generalized model remains tractable non-numerically.

An alternative approach capable of producing some of the metrics in a friendlier sparse-matrix form will be then conjectured and studied in Sections 6 and 7. In the former section we shall put more emphasis upon the phenomenological aspects of the metrics reflected by specific choices of optional