Annals of Physics 361 (2015) 303–329

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

From doubled Chern–Simons–Maxwell lattice gauge theory to extensions of the toric code

T.Z. Olesen [a,*], N.D. Vlasii [a,b], U.-J. Wiese [a]

[a] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, Bern University, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

[b] *Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, 14-b Metrologichna Str., Kyiv, 03680, Ukraine*

ARTICLE INFO

Article history:
Received 20 April 2015
Accepted 19 June 2015
Available online 8 July 2015

Keywords:
Quantum information
Toric code
Chern–Simons theory
Lattice gauge theory
Self-adjoint extension
Berry gauge field

ABSTRACT

We regularize compact and non-compact Abelian Chern–Simons–Maxwell theories on a spatial lattice using the Hamiltonian formulation. We consider a doubled theory with gauge fields living on a lattice and its dual lattice. The Hilbert space of the theory is a product of local Hilbert spaces, each associated with a link and the corresponding dual link. The two electric field operators associated with the link-pair do not commute. In the non-compact case with gauge group $\mathbb{R}$, each local Hilbert space is analogous to the one of a charged "particle" moving in the link-pair group space $\mathbb{R}^2$ in a constant "magnetic" background field. In the compact case, the link-pair group space is a torus $U(1)^2$ threaded by k units of quantized "magnetic" flux, with k being the level of the Chern–Simons theory. The holonomies of the torus $U(1)^2$ give rise to two self-adjoint extension parameters, which form two non-dynamical background lattice gauge fields that explicitly break the manifest gauge symmetry from $U(1)$ to $\mathbb{Z}(k)$. The local Hilbert space of a link-pair then decomposes into representations of a magnetic translation group. In the pure Chern–Simons limit of a large "photon" mass, this results in a $\mathbb{Z}(k)$-symmetric variant of Kitaev's toric code, self-adjointly extended by the two non-dynamical background lattice gauge fields. Electric charges on the original lattice and on the dual

* Corresponding author.
E-mail addresses: olesen@itp.unibe.ch (T.Z. Olesen), vlasii@itp.unibe.ch (N.D. Vlasii), wiese@itp.unibe.ch (U.-J. Wiese).

http://dx.doi.org/10.1016/j.aop.2015.06.020

lattice obey mutually anyonic statistics with the statistics angle $\frac{2\pi}{k}$. Non-Abelian $U(k)$ Berry gauge fields that arise from the self-adjoint extension parameters may be interesting in the context of quantum information processing.

1. Introduction

Gauge theories in two spatial dimensions may contain a Chern–Simons term [1–5] which explicitly breaks parity and time-reversal symmetry. In these theories the gauge field acquires a mass and the charged particles obey fractional anyonic statistics [6–9]. Non-Abelian Chern–Simons theories have intricate relations to knot theory, the Jones polynomials [10], and to 2-dimensional conformal field theories [11,12]. Abelian Chern–Simons theories have been used to facilitate bosonization or Fermi–Bose transmutation in $(2 + 1)$ dimensions [13–16]. Furthermore, Chern–Simons gauge theories are of central importance in the context of the fractional quantum Hall effect [17–19] and other condensed matter systems [20–26]. They also play an important role for topological quantum computation [27–32]. By braiding world-lines of anyonic quasi-particles, one can accumulate appropriate non-Abelian Berry phases [33,34], which can encode quantum information. When the anyons obey a sufficiently complex version of non-Abelian braid statistics, they can be employed to realize the quantum gates that are sufficient to realize universal quantum computation [35–37]. The idea of topological quantum computation is attractive because the quantum information is then naturally protected from decoherence by the topological nature of the non-Abelian Berry phases. In particular, information is not stored locally but is distributed throughout the entire system. The toric code is a $(2 + 1)$-d $\mathbb{Z}(2)$ lattice gauge theory which can be used as a topologically protected storage device for quantum information [27]. This theory has charges and dual charges with mutually anyonic statistics.

In this paper, we derive extensions of the toric code from a doubled compact lattice Chern–Simons–Maxwell theory with Abelian gauge group $U(1)$. Similar to fermions, topologically massive gauge fields also suffer from a lattice doubling problem. Here we are not trying to circumvent this problem but work with a lattice gauge field and an independent gauge field associated with the dual lattice. The fundamental gauge degrees of freedom are then associated with a cross formed by a link and its corresponding dual link. The Chern–Simons term couples the two lattices and implies that the canonically conjugate momenta (i.e. the electric field strengths) of the original and the dual gauge field do not commute. In ordinary lattice gauge theories [38–41] the field algebra is link-based. This framework has also been used in studies of Chern–Simons gauge theories on the lattice [14,42–45]. In our lattice formulation of doubled Chern–Simons gauge theories, on the other hand, the field algebra is cross-based. Such a system was already investigated in [46,47]. Here we concentrate on the relation of this theory to the toric code.

In ordinary lattice gauge theory with a link-based field algebra every link has a "mechanical" analog. It behaves like a "particle" moving in the group space. For example, the dynamics of a link variable in a compact Abelian $U(1)$ lattice gauge theory is analogous to the one of a quantum mechanical particle moving on a circle. Similarly, in our unconventional lattice gauge theory with a cross-based field algebra, the "mechanical" analog of each cross is a charged "particle" moving on a 2-dimensional group space torus $U(1)^2$ threaded by an abstract "magnetic" field [48–50]. The Dirac quantization condition for the abstract "magnetic" flux then implies the quantization of the level k — the prefactor of the Chern–Simons term. Interestingly, the corresponding cross-based Hamiltonian has two self-adjoint extension parameters, which naturally enter the quantum theory as external parameters, while the classical theory is insensitive to these parameters. Remarkably, the self-adjoint extension parameters (which are associated with the links and the dual links) themselves form two non-dynamical $U(1)$ lattice gauge fields. This reduces the manifest gauge symmetry of the quantum