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Fine structure in the large n limit of the non-Hermitian Penner matrix model

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In this paper we apply results on the asymptotic zero distribution of the Laguerre polynomials to discuss generalizations of the standard large n limit in the non-Hermitian Penner matrix model. In these generalizations $g_n n \to t$, but the product $g_n n$ is not necessarily fixed to the value of the 't Hooft coupling t. If $t > 1$ and the limit $l = \lim_{n\to\infty} |\sin(\pi/g_n)|^{1/n}$ exists, then the large n limit is well-defined but depends both on t and on l. This result implies that for $t > 1$ the standard large n limit with $g_n n = t$ fixed is not well-defined. The parameter l determines a fine structure of the asymptotic eigenvalue support: for $l \neq 0$ the support consists of an interval on the real axis with charge fraction $Q = 1 - 1/t$ and an l-dependent oval around the origin with charge fraction $1/t$. For $l = 1$ these two components meet, and for $l = 0$ the oval collapses to the origin. We also calculate the total electrostatic energy $\mathcal{E}$, which turns out to be independent of l, and the free energy $\mathcal{F} = \mathcal{E} - Q \ln l$, which does depend on the fine structure parameter l. The existence of large n asymptotic expansions of $\mathcal{F}$ beyond the planar limit as well as the double-scaling limit are also discussed.

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1. Introduction

The large n limit with fixed 't Hooft coupling $n4\pi g_{\mathrm{YM}}^2 = \lambda$ of non-abelian gauge theories has been the subject of intensive research for more than four decades, and in particular has fostered the study of large n matrix field theories. Random matrix models are a simplified version of these theories that offer a convenient setting to explore large n limits, because the partition function of a $n \times n$ random matrix model can be written as an integral over the matrix eigenvalues z_i:

$$Z_n(g) = \frac{1}{n!} \int_{\Gamma\times\cdots\times\Gamma} \prod_{j<k} (z_j - z_k)^2 \exp\left(-\frac{1}{g}\sum_{i=1}^{n} W(z_i)\right) \prod_{i=1}^{n} \mathrm{d}z_i. \tag{1}$$

The standard large n limit is defined here as $n \to \infty$ with fixed 't Hooft coupling $t = ng$. This limit exhibits a number of interesting properties in the case of Hermitian matrix models with polynomial potentials $W(z)$, such as the existence of an asymptotic eigenvalue density supported on a finite number of real intervals (cuts), and the existence of a topological expansion of the free energy in the one-cut case [1–4].

In this paper we discuss this large n limit and generalizations thereof for the Penner matrix model with potential

$$W(z) = z + \log z. \tag{2}$$

The logarithmic function in (2) is defined by $\log z = \ln|z| + \mathrm{i}\arg z$ with $0 \leq \arg z < 2\pi$ and where $\ln$ denotes the real logarithmic function on the positive real axis. We consider the Penner model on any path Γ in (1) homotopic in $\mathbb{C} \setminus [0, +\infty)$ to the steepest descent path $\operatorname{Im} W(z) = \pi$ through the critical point $z = -1$ of $W(z)$ illustrated in Fig. 1. Obviously, the corresponding partition function (1) cannot be interpreted as an integral over the (real) eigenvalues of a Hermitian matrix model, but to a model in which the integration is performed over a set of $n \times n$ complex matrices with n eigenvalues constrained to lie on the path Γ. In the literature [5,6] these models are frequently called *holomorphic* matrix models. However, to emphasize the relation of (1) to the theory of non-Hermitian Laguerre orthogonal polynomials, hereafter we call our model the non-Hermitian Penner model.

The Hermitian Penner model was introduced in [7] because the large n expansion of its free energy provided generating functionals for the Euler characteristics $\chi_{k,s}$ of the moduli spaces of Riemann surfaces of genus k with $s \geq 1$ punctures. It was found later [8] that the analytic continuation of the free energy in the Hermitian case to the non-Hermitian case and a suitable double scaling limit yielded a generating functional for the Euler characteristics of unpunctured Riemann surfaces too. As a consequence, the Penner model is closely related to the $c = 1$ noncritical string with one direction of the spacetime compactified on the circle with self-dual radius [8–12]. More recently, the model has been used to analyze nonperturbative effects in $c = 1$ string theory and matrix models [13,14].

In the early 90's Tan [10,11], and Ambjørn, Kristjansen and Makeenko [12] applied the saddle point method to give what was thought to be a complete description of the large n limit of the Penner model for both positive and negative fixed values of the 't Hooft coupling.

The present paper is motivated by the behavior of the asymptotic zero distribution of the Laguerre polynomials with negative parameter discovered by Kuijlaars and McLaughlin [15]. Note that fixing the 't Hooft coupling t in effect defines a sequence

$$g_n = \frac{t}{n} \tag{3}$$

which trivially satisfies

$$\lim_{n\to\infty} n g_n = t. \tag{4}$$

We will show that the behavior discovered by Kuijlaars and McLaughlin [15] also arises in the large n limit of the non-Hermitian Penner model if the sequence g_n is not restricted to be (3), but only required to have the limit (4) with $t > 1$ and to have a finite value l for the following limit:

$$l = \lim_{n\to\infty} |\sin(\pi/g_n)|^{1/n}. \tag{5}$$