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Yangian-invariant spin models and Fibonacci numbers

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HIGHLIGHTS

- Exact computation of the minimum average degeneracy of Yangian-invariant spin models.
- Applications to spin chains of Haldane–Shastry type and 1-d vertex models.
- Connections with generalized Fibonacci numbers and 1-d anyons.

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ABSTRACT

We study a wide class of finite-dimensional $\mathrm{su}(m|n)$-supersymmetric models closely related to the representations of the Yangian $Y(\mathrm{gl}(m|n))$ labeled by border strips. We quantitatively analyze the degree of degeneracy of these models arising from their Yangian invariance, measured by the average degeneracy of the spectrum. We compute in closed form the minimum average degeneracy of any such model, and show that in the non-supersymmetric case it can be expressed in terms of generalized Fibonacci numbers. Using several properties of these numbers, we show that (except in the simpler $\mathrm{su}(1|1)$ case) the minimum average degeneracy grows exponentially with the number of spins. We apply our results to several well-known spin chains of Haldane–Shastry type, quantitatively showing that their degree of degeneracy is much higher than expected for a generic Yangian-invariant spin model. Finally, we show that the set of distinct levels of a Yangian-invariant spin model is described by an effective model of quasi-particles. We study this effective model, discussing

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its connections to one-dimensional anyons and properties of generalized Fibonacci numbers.

1. Introduction

In this paper we shall consider a general class of finite-dimensional quantum models which by construction are invariant under the Yangian algebra $Y(\mathrm{gl}(m|n))$. The key to the Yangian invariance of these models is their connection to certain finite-dimensional representations of $Y(\mathrm{gl}(m|n))$ labeled by a type of skew Young diagrams, the so called border strips [1,2]. These representations play a fundamental role in the study of the integrable two-dimensional conformal field theory related to the latter models, namely the $\mathrm{su}(m|n)$ WZNW model at level 1 [3–5]. The models we shall be interested in, which we shall refer to as Yangian-invariant $\mathrm{su}(m|n)$ spin models, are characterized by the fact that their Hilbert space is a direct sum of the irreducible representations of $Y(\mathrm{gl}(m|n))$ labeled by border strips with exactly N boxes, where N is the number of spins. The simplest examples of these models are certain integrable spin chains with long range interactions invariant under the Yangian for a finite number of spins, namely the $\mathrm{su}(m|n)$ supersymmetric versions of the Haldane–Shastry [6–8] and Polychronakos–Frahm [9–11] chains; see, e.g., [1,3,5,12,13].

As is well known, the Haldane–Shastry (HS) spin chain is a circular (translationally invariant) chain with equally spaced sites, the interactions between any two spins being inversely proportional to the square of their chord distance. This model appears in connection with a wide variety of topics in theoretical and mathematical physics, including one-dimensional anyons [3,5,8,14], conformal field theory [15–18], quantum chaos [19–22], quantum information theory [23], and quantum integrability [12,24,25]. A distinctive feature of the HS chain is the fact that it can be obtained from a dynamical model, namely the spin Sutherland (trigonometric) model [26–28], in the strong coupling limit [9]. By applying the same procedure to the spin Calogero (rational) [29,30] and Inozemtsev (hyperbolic) [31] models, one obtains the Polychronakos–Frahm (PF) and Frahm–Inozemtsev (FI) [32] spin chains, which share many properties with the original HS chain. In fact, although the Yangian symmetry of the FI chain has not been rigorously established, we shall see in Section 2 that it is isospectral to a Yangian-invariant spin model.

It is clear by its very definition that the class of Yangian-invariant spin models encompasses a wide range of quantum systems. For instance, apart from the integrable spin chains mentioned above, it includes the family of one-dimensional vertex models studied in Refs. [13,33]. A common feature shared by all Yangian-invariant spin models is the high degeneracy of their spectrum, stemming from the Yangian symmetry. This statement, however, is far from precise, and does not shed any light on whether a certain model in this class has additional degeneracy due to its particular form. The main aim of this paper is precisely to perform a quantitative analysis of the degeneracy inherent to all Yangian-invariant spin models. To this end, we shall use as a concrete measure of the degree of degeneracy of a finite-dimensional quantum system its average degeneracy, defined as the quotient of the dimension of its Hilbert space by the number of distinct energy levels. We shall derive closed-form expressions for the average degeneracy of a "generic" Yangian-invariant $\mathrm{su}(m|n)$ spin model, both in the supersymmetric ($mn \neq 0$) and non-supersymmetric cases. In fact, these expressions provide a lower bound for the average degeneracy of *any* Yangian-invariant spin model, and thus can be used as a practical test for ruling out that a particular quantum system belongs to this class.

As it turns out, the behavior of this lower bound (which we have termed "minimum average degeneracy") is rather different in the supersymmetric and non-supersymmetric cases. In the former case, the minimum average degeneracy is given by a simple power law in terms of the dimension $m+n$ of the one-particle Hilbert space (cf. Eq. (17)). Far more interestingly, in the non-supersymmetric case the minimum average degeneracy is expressed in terms of generalized Fibonacci numbers [34], which reduce to the standard ones for spin 1/2 (m or n equal to 2). Using standard properties of the