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Restricted phase-space approximation in real-time stochastic quantization

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ABSTRACT

We perform and extend real-time numerical simulation of a low-dimensional scalar field theory or a quantum mechanical system using stochastic quantization. After a brief review of the quantization method and the complex Langevin dynamics, we calculate the propagator and make a comparison with analytical results. This is a first step toward general applications, and we focus only on the vacuum properties of the theory; this enables us to handle the boundary condition with the $i\epsilon$ prescription in frequency space. While we can control stability of the numerical simulation for any coupling strength, our results turn out to flow into an unphysical fixed-point, which is qualitatively understood from the corresponding Fokker–Planck equation. We propose a simple truncation scheme, "restricted phase-space approximation", to avoid the unphysical fixed-point. With this method, we obtain stable results at reasonably good accuracy. Finally we give a short discussion on the closed-time path formalism and demonstrate the direct computation of the vacuum expectation value not with the $i\epsilon$ prescription but from an explicit construction of the Feynman kernel.

1. Introduction

Large-scale numerical computation is becoming a vital building block in today's scientific researches. In theoretical physics, the numerical approach is regarded as a starting point of a pursuit

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toward fundamental understanding of new phenomena. Performing numerical *experiments*, we can test ideas and hypotheses in an ideal setup repeatedly and easily, which is usually difficult in real experiments. This enables us to efficiently build models and theories that describe nature. In this spirit, in order to study new physics, it is important to develop new numerical methods and extend their validity.

Quantum field theories that accommodate infinite degrees of freedom stand in the center of modern physics. It is becoming less and less costly to perform large-scale numerical simulations thanks to tremendous developments in the computing power and the various innovations in the numerical algorithms. One area where computers are playing an important role is the fundamental theory of the strong interaction; that is, quantum chromodynamics (QCD) [1] can be formulated on the four-dimensional lattice grid in Euclidean space–time, so that the exponentiated action, $e^{-S_{QCD}}$, is a real positive number and can be interpreted as a weight factor in analogy with statistical mechanics [2]. We can then carry out the functional integral by means of the Monte-Carlo algorithm as long as the weight factor is real and non-negative. This approach known as the lattice-QCD simulation [3] has been the most successful non-perturbative tool to investigate the QCD-vacuum (topological) structure [4], thermodynamics of QCD matter [5,6], the hadron spectroscopy [7], and also the real-time characters such as the spectral function [8,9], the particle production rate [10,11], and the transport coefficients [12–16], etc.

Another area where numerical simulations are intensively utilized is condensed matter physics, especially in the field of strongly correlated electron systems. It has been realized that quantum many-body effect leads to various phase transitions. A well studied example is the Mott transition [17] in which electrons freeze their motion due to strong Coulomb interaction. It is believed that this transition is relevant to understanding of the pairing mechanism of high temperature superconductivity [18]. Numerical algorithms such as the density matrix renormalization group (DMRG) [19,20] and the dynamical mean field theory (DMFT) [21] have been developed and applied to problems in correlated electron systems. Recently, real-time dynamics in condensed matter systems is becoming a hot topic (for a review see Ref. [22]). One important problem is the physics of quantum quench; i.e., many-body dynamics triggered by a rapid parameter change. Phenomena such as "prethermalization" [23], initially discovered in the QCD community, has motivated many condensed matter researches [24,25]. In condensed matter, variety of methods exist to deal with the problem of a time-evolving quantum many-body system. They range from direct wave-function-based techniques such as exact diagonalization and DMRG [26–28], quantum master equations [29], and quantum kinetic equations [30], to the Keldysh formalism for the non-equilibrium Greens functions [31–33]. Using the Keldysh formalism, many sophisticated theoretical techniques that were developed for equilibrium can be straightforwardly utilized to non-equilibrium systems; e.g., diagrammatic quantum Monte-Carlo method (QMC) [34,35] and non-equilibrium DMFT [22,36]. The price to pay is the severe negative sign problem, and it is still challenging to study the long time behavior. We summarize major approaches for non-equilibrium many-body systems in Table 1.

In order to study non-linear QCD processes far from equilibrium such as the pattern formation [37] and the turbulent flow [38], a method that can treat not only fermions but also bosons must be developed. To overcome the limitation of the Monte-Carlo simulation, some alternative approaches are proposed such as the gauge/gravity correspondence [39], the classical statistical field theory [40–42], the 2-particle-irreducible formalism [43] (see also Ref. [31]), and the stochastic quantization [44–47].

The gauge/gravity correspondence has provided us with useful insights into the thermalization problem and the numerical simulations are possible now to trace the evolution processes of the dynamical system [48–54], though the technique can be applied only to a special class of the strong-coupling gauge theory. The classical statistical simulation, which is also known as the "truncated Wigner" approximation [55], is quite successful in describing the early stages of the relativistic heavy-ion collision [56–59], which has been closely investigated in connection to the wave turbulence and the scaling behavior also [60–68].

Although the classical statistical simulation is a useful tool in the regime where the occupation number is large enough to justify the classical treatment, the formalism itself needs to be elaborated not to ruin the renormalizability [69]. For this purpose it is an interesting question to think of a possible relation between the classical statistical approach and stochastic quantization as speculated