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Nonlinear dynamics of classical counterpart of the generalized quantum nonlinear oscillator driven by position dependent mass

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HIGHLIGHTS

- Mass parameter η plays a crucial role in the chaotic dynamics of the system.
- For $\eta > 0$, period doubling route to chaos occurs.
- For $\eta < 0$, quasiperiodic route to chaos occurs via strange nonchaotic attractor.
- Period five orbit occurs after period four for fixed $\eta > 0$ when other parameters vary.
- Fractal boundaries are observed in the $f - \alpha$ bifurcation diagram.

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ABSTRACT

This paper examines the chaotic dynamics of certain damped and forced versions of classical counterpart of generalized quantum nonlinear oscillator endowed with position dependent mass (PDM). Various bifurcations such as symmetry breaking, period doubling, inverse period doubling, interior and boundary crises are reported. Sensitivity of the mass parameter η to the chaotic dynamics of the system is demonstrated by the appearance of completely different route to chaos for $\eta > 0$ and $\eta < 0$. In the former case the chaotic motion is found to set in through period doubling route while in the latter case there is quasiperiodic route to chaos via strange non-chaotic attractor. Fractal boundaries are observed in chaos plots for $\eta > 0$.

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1. Introduction

A nonlinear oscillator having simple harmonic periodic motions [1] is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{1+\eta x^2}\right)(\dot{x}^2 - \zeta x^2) \tag{1}$$

where dot denotes derivative with respect to time t and $\zeta\,(>0)$, $\eta(\neq 0)$ are parameters. The Lagrangian (1) is the single particle analogue of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\frac{(\partial_\mu \phi)^2 - m^2\phi^2}{1+\eta\phi^2} \tag{2}$$

of a relativistic scalar field. This type of nonpolynomial Lagrangians were considered in the formulation of chirally invariant Lagrangian models in particle physics [2].

It is important to note that in (1) the parameter η is present not only in the potential $\frac{x^2}{1+\eta x^2}$ but also in the kinetic term. So this nonlinear oscillator represents a system with a position dependent mass [3]. When $\eta < 0$, the dynamics is restricted to the interval $-\frac{1}{\sqrt{|\eta|}} < x < \frac{1}{\sqrt{|\eta|}}$ while for $\eta > 0$, there is no such restriction. Recently this particular nonlinear oscillator has been generalized to the higher dimensions and various properties of this system has been studied [4]. The classical Hamiltonian corresponding to this oscillator is given by

$$H = \frac{1}{2}[p^2(1+\eta x^2) + \zeta x^2(1+\eta x^2)^{-1}] \tag{3}$$

where $p = \frac{\dot{x}}{(1+\eta x^2)}$. The Euler–Lagrange equation of motion derived from (1) is given by

$$(1+\eta x^2)\ddot{x} - \eta x\dot{x}^2 + \zeta x = 0. \tag{4}$$

This equation of motion represents the motion of a particle endowed with position dependent mass $m(x) = \frac{1}{1+\eta x^2}$ subjected to move in an oscillator potential $V(x) = \frac{1}{2}m(x)\zeta x^2$ [5]. The oscillator system (4) exhibits simple harmonic periodic solutions but with amplitude dependent frequency

$$x(t) = A\cos(\Omega t + \delta), \quad \Omega = \frac{\sqrt{\zeta}}{\sqrt{1-\eta A^2}}. \tag{5}$$

It is to be mentioned that this dependence can become highly sensitive when suitable additional external force, nonlinearity, damping, etc. are added, leading to bifurcation and chaos [6]. The nonlinear equation (3) is of Lienard II type: $\ddot{x}^2 + f(x)\dot{x}^2 + g(x) = 0$ [6,7] involving a quadratic damping term in contrast to the usual Lienard equation $\ddot{x}^2 + f(x)\dot{x} + g(x) = 0$. Such equations with a quadratic damping term often result from the movement of an object through a fluid medium. For a practical application one may note that such quadratic damping assumes significance in the design of a cam, which is an eccentric shaped device usually used to convert the rotational motion of a shaft into a translational one especially at high speeds or in case of very dense fluids [8]. On the other hand it can be shown that the Lienard II equation arises whenever the mass is position dependent [9].

The quantization of this nonlinear oscillator was done in [10,11] and the corresponding position dependent mass Schrödinger equation (PDMSE) with $\hbar = 1$ becomes

$$\left[-\frac{1}{2m}(1+\eta x^2)\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)\eta x\frac{d}{dx} + \frac{1}{2}g\frac{x^2}{1+\eta x^2}\right]\psi = E\psi \tag{6}$$

where $g = \zeta(m\zeta + \eta)$. It may be pointed out that this η-dependent system can be considered as a deformation of the standard harmonic oscillator in the sense that for $\eta \to 0$ all the characteristics of the linear harmonic oscillator are recovered. In Ref. [10,11], the PDMSE corresponding to this nonlinear oscillator has been solved exactly as a Sturm Liouville problem and η-dependent eigenvalues and eigenfunctions were obtained for both $\eta > 0$ and $\eta < 0$ which are given below: