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Non-Hermitian systems of Euclidean Lie algebraic type with real energy spectra

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HIGHLIGHTS

- Different PT-symmetries lead to qualitatively different systems.
- Construction of non-perturbative Dyson maps and isospectral Hermitian counterparts.
- Numerical discussion of the eigenvalue spectra for one of the E(2)-systems.
- Established link to systems studied in the context of optical lattices.
- Setup for the E(3)-algebra is provided.

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ABSTRACT

We study several classes of non-Hermitian Hamiltonian systems, which can be expressed in terms of bilinear combinations of Euclidean–Lie algebraic generators. The classes are distinguished by different versions of antilinear (PT)-symmetries exhibiting various types of qualitative behaviour. On the basis of explicitly computed non-perturbative Dyson maps we construct metric operators, isospectral Hermitian counterparts for which we solve the corresponding time-independent Schrödinger equation for specific choices of the coupling constants. In these cases general analytical expressions for the solutions are obtained in the form of Mathieu functions, which we analyze numerically to obtain the corresponding energy spectra. We identify regions in the parameter space for which the corresponding spectra are entirely real and also domains where the PT symmetry is spontaneously broken and sometimes also regained at exceptional points. In some cases it is shown explicitly how the threshold region from real to complex spectra is characterized by the breakdown of the Dyson maps or the metric

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operator. We establish the explicit relationship to models currently under investigation in the context of beam dynamics in optical lattices.

1. Introduction

Quasi-exactly solvable models [1] of Lie algebraic type are believed to be almost all related to $sl_2(\mathbb{C})$ with their compact and non-compact real forms $su(2)$ and $su(1,1)$, respectively [2]. The nature of those models dictates that essentially all the wavefunctions related to solutions for the time-independent Schrödinger equation of these type of models may be expressed in terms of hypergeometric functions. Non-Hermitian variants of these models expressed generically in terms of $su(2)$ or $su(1,1)$ generators have been investigated systematically in [3,4] and large classes of models were found to possess real or partially spectra despite their non-Hermitian nature. Under certain constraints on the coupling constants the models could be mapped to Hermitian isospectral counterparts. Positive Hermitian metric operators were shown to exist, such that a consistent quantum mechanical description of these models is possible when following the general techniques developed over the last years [5–7] in the context of $\mathcal{PT}$-symmetric non-Hermitian quantum mechanics.

It is, however, also well known that there exists an interesting subclass of solvable models related to Mathieu functions which are known to possess solutions, which are not expressible in terms of hypergeometric functions. In a more generic setting these type of models are known to be related to specific representations of the Euclidean algebra rather than to its subalgebra $sl_2(\mathbb{C})$. This feature makes models based on them interesting objects of investigation from a mathematical point of view. In a more applied setting it is also well known that the Mathieu equation arises in optics as a reduction from the Helmholtz equation. This analogue setting of complex quantum mechanics is currently under intense investigation. Concrete versions of complex potentials leading to real Mathieu potentials have recently been studied from a theoretical as well as experimental point of view in [8–13]. Further applications are found for instance in the investigation of complex crystals [14].

It was recently shown that for E_2 [15] and E_3 [16] some simple non-Hermitian versions also possess real spectra. Here we will follow the line of thought of [3] and investigate systematically the analogues of quasi-exactly solvable models of Lie algebraic type, that is those models which can be written as bilinear combinations in terms of the Euclidean algebra generators.

Our manuscript is organized as follows: At the beginning of Section 2 we discuss five different types of $\mathcal{PT}$-symmetries for the E_2-algebra and present the computation of the adjoint action on their generators. In the following five subsection we derive Dyson maps and isospectral counterparts for generic non-Hermitian Hamiltonians invariant under these different types of symmetries. For the last symmetry we present a more detailed analysis of the time-independent Schrödinger equation. We derive some explicit analytical solutions, which we analyze numerically to compute the corresponding energy spectra leading to three qualitatively different scenarios: entirely real energies, spectra with spontaneously broken $\mathcal{PT}$-symmetry at exceptional points characterized by two or three disconnected regions in the parameter space. We propose a measurable quantity that can be used as a criterion to identify the spontaneously broken $\mathcal{PT}$-symmetric regime. In Section 3 we discuss the $\mathcal{PT}$-symmetries for the E_3-algebra, present the computation of the adjoint action on its generators and indicate how to obtain simple examples of explicit isospectral pairs of an E_3-invariant non-Hermitian and Hermitian Hamiltonian.

2. $\mathcal{PT}$-symmetric E_2-invariant non-Hermitian Hamiltonians

We take here the commutation relations obeyed by the three generators u, v and J as the defining relations of the Euclidean-algebra E_2

$$[u, J] = iv, \qquad [v, J] = -iu, \quad \text{and} \quad [u, v] = 0. \tag{2.1}$$