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Superintegrable systems on spaces of constant curvature

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HIGHLIGHTS

- Classifying 2D superintegrable, separable (polar coordinates) systems on S^2, R^2, H^2.
- Construction of radial, angular potentials leading to superintegrability.
- Generalization of Bertrand's theorem covering known models, e.g. Higgs, TTW, PW, and Coulomb.

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ABSTRACT

Construction and classification of two-dimensional (2D) superintegrable systems (i.e. systems admitting, in addition to two global integrals of motion guaranteeing the Liouville integrability, the third global and independent one) defined on 2D spaces of constant curvature and separable in the so-called geodesic polar coordinates are presented. The method proposed is applicable to any value of curvature including the case of Euclidean plane, sphere and hyperbolic plane. The main result is a generalization of Bertrand's theorem on 2D spaces of constant curvature and covers most of the known separable and superintegrable models on such spaces (in particular, the so-called Tremblay–Turbiner–Winternitz (TTW) and Post–Winternitz (PW) models which have recently attracted some interest).

1. Introduction

A system with n degrees of freedom possessing n global and functionally independent integrals of motion in involution is by the definition integrable in the Liouville sense [1].

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The integrable systems admitting more (global and functionally independent) integrals of motion than degrees of freedom are called superintegrable [2]. If they have maximal possible number of independent constants, i.e. $2n-1$ they are called maximally superintegrable. The Kepler model and isotropic harmonic oscillator provide the canonical examples of such systems. The two extra integrals of motion in these models which do not arise from an explicit geometrical invariance of the potential are constructed out of the famous Runge–Lenz vector [3] in case of the Kepler system and so-called Fradkin tensor [4] in the oscillator case. The additional constants together with the Liouville ones generate a higher symmetry. It is $SO(4)$ group (when restricted to the sub-manifold of constant energy) in the case of Kepler problem [5] and $SU(3)$ group in the isotropic harmonic oscillator case [6].

If n-dimensional submanifold of phase space determined by the n involutive first integrals is compact and connected it is topologically equivalent to the n-dimensional so-called Arnold–Liouville torus (in general non-compact case, it is the product of a torus and Euclidean space) [1]. Now, due to the existence of additional constants of motion the trajectories of superintegrable systems are restricted to lower dimensional submanifolds of Arnold–Liouville tori. In the particular case of maximally superintegrable systems, when the number of global independent integrals of motion increases to $2n-1$, the classical trajectories are closed curves. In fact, the property that all bounded trajectories are closed can be regarded as an equivalent definition of maximal superintegrability [1]. From this point of view, the old and very elegant Bertrand's theorem [7] which states that the only central potentials for which all bounded trajectories are closed are just Kepler and isotropic oscillator ones provides a complete classification of 3-D superintegrable systems with central potentials.

In general, a dynamics in non-central potentials is much more complicated than in the central ones. Consequently, a search for superintegrable systems in non-central fields is more involved. However, there is a number of papers devoted to the study of the superintegrability in non-central potentials, both in the Euclidean and curved configuration spaces [8–26].

Many years ago, Onofri and Pauri managed to classify, on general ground, all superintegrable systems defined on 2D Euclidean plane with Hamiltonians separable in the polar coordinates [27]. Unfortunately, it seems that this very nice and interesting result is not as well known as it deserves to be. This is perhaps due to a rather involved method of derivation the authors used.

In the present paper we discuss a construction and classification of 2D superintegrable systems defined on spaces of constant curvature and separable in the so-called geodesic polar coordinates. Our method works for any value of the curvature including the case of Euclidean plane, the sphere and the hyperbolic plane.

The paper is organized as follows. In Section 2 we set our notation and explain the main task, that is a construction and classification of radial and angular potentials leading to superintegrable dynamics.

A necessary and sufficient condition for the integrable system to be superintegrable is recalled in Section 3. In the framework of the technique of action-angle variables it states that the Hamiltonian of the superintegrable system has to be a function of a linear combination of action variables with integer coefficients. Then we show that this condition, when applied to a 2D integrable and separable system, can be formulated in the form of an equality (up to an integer factor) of radial and angular periods of motion. The radial period corresponds to the dynamics in an isochronous (i.e. such that the period of motion does not depend on energy) effective potential being determined by the radial potential entering the original Hamiltonian. This simple consequence of the superintegrability condition plays a key role in our method. First, it implies that the search for our superintegrable systems can be started with the construction of the effective isochronous radial potentials (actually its $\tilde{V}_\sigma(\rho)$ part (see below) being directly related to the radial potential $V_\sigma(r)$ entering the original Hamiltonian). This is done in Section 4, where the relevant equation for $\tilde{V}_\sigma(\rho)$ has been introduced and solved. Knowing $\tilde{V}_\sigma(\rho)$ potentials allows us to find the original radial potentials $V_\sigma(r)$ leading to superintegrable systems. Next, having these potentials we calculate the periods of angular motions. Finally, considering the formula for a period of one-dimensional motion in a potential as the integral equation (actually it is the equation of Abel's type) we find the angular potentials corresponding to the periods of angular motions.

Section 4 contains also the discussion of the explicit forms of the corresponding radial action variables. It is explained how these actions and the superintegrability condition can be used to find the explicit