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Natural star-products on symplectic manifolds and related quantum mechanical operators

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HIGHLIGHTS

- Invariant representations of natural star-products on symplectic manifolds are considered.
- Star-products induced by flat and non-flat connections are investigated.
- Operator representations in Hilbert space of considered star-algebras are constructed.

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ABSTRACT

In this paper is considered a problem of defining natural star-products on symplectic manifolds, admissible for quantization of classical Hamiltonian systems. First, a construction of a star-product on a cotangent bundle to an Euclidean configuration space is given with the use of a sequence of pair-wise commuting vector fields. The connection with a covariant representation of such a star-product is also presented. Then, an extension of the construction to symplectic manifolds over flat and non-flat pseudo-Riemannian configuration spaces is discussed. Finally, a coordinate free construction of related quantum mechanical operators from Hilbert space over respective configuration space is presented.

1. Introduction

The formalism of quantization of systems described by configuration spaces in the form of Euclidean spaces is well established and confirmed by experiments. The next step should be theory of quantization of systems defined on curved spaces, e.g. systems with constraints or systems coupled

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with classical gravitational fields. This task however constitutes some problems as, because of the lack of experiments, it is difficult to find a proper generalization of the quantization formalism. The only thing one can do is to work on the mathematical level and try to find some distinguished quantization schemes with interesting properties from the wast number of possibilities.

This paper aims in a discussion of this problem from a point of view of deformation quantization theory. In this approach to quantum mechanics the quantization is basically given by introducing a star-product on a phase space. Thus in this paper we will deal first with a problem of defining natural star-products on symplectic manifolds (phase spaces), and second with their appropriate operator representation in a Hilbert space over configuration space.

In the work of Bayen et al. [1,2] there was presented a construction of a star-product on a symplectic manifold endowed with a flat symplectic linear connection. Later Fedosov [3] presented a construction of an admissible star-product for a general symplectic connection. The resulting star-products were given in a covariant form independent on the coordinate system. It should be noted that Kontsevich [4] proved the existence of a star-product on a general Poisson manifold, not necessary a symplectic one. These results, although elegant, are difficult to use in computations. In this paper first we discuss an alternative way of introducing a star-product on symplectic manifold. It is base on a definition of a star-product with the use of a maximal sequence of pair-wise commuting vector fields. In this way equations for star-products are of simpler form and can be easier used in computations. Moreover, we discuss the connection of the vector field representation of the star-product to the covariant form of the star-product (Section 2).

An important property of the star-product is an equivalence with the Moyal product. This allows introduction of the operator approach to quantum mechanics [5,6]. It is known how to pass to the operator representation of quantum mechanics in the case of Euclidean configuration spaces. In a general case we can use the fact that for any classical and quantum canonical coordinate system the star-product is equivalent with the Moyal product. This property allows to construct the operator representation of quantum mechanics from the knowledge of this construction in Euclidean case. In Section 3 we construct the equivalence for the star-product written in a covariant form on a flat symplectic manifold.

In Section 4 we discuss how to introduce star-products on a general symplectic manifold in a natural manner. We also present an example of such products, which construction involves symplectic linear connection on a symplectic manifold.

Section 5 is devoted to a problem of associating to star-algebras certain algebras of operators defined on particular Hilbert spaces. Usually, in the literature, one can find this connection for a Moyal star-product written in Cartesian coordinates. The general case seems not to be considered yet. Worth noting is the paper [7] where the authors present an interesting approach to the star-gen value equation and its relation to the ordinary Hilbert space approach, based on the theory of pseudo-differential operators. We describe a connection between star-algebras and respective operator algebras for a very general family of star-products considered in the paper. In particular we describe a procedure of associating, in a coordinate independent way, to every phase space function an operator defined on a Hilbert space of square integrable functions defined on a configuration space. We also give examples of operators linear, quadratic and cubic in momenta written in an invariant form and derived for a very general star-product defined on a symplectic manifold over a curved pseudo-Riemannian space.

In Section 6 are made some remarks about quantization of classical Hamiltonian systems. We also discuss a problem, using the results presented in the paper, of choosing a physically admissible quantizations for Hamiltonian systems from phase spaces considered in the paper.

2. The case of a symplectic manifold T^*E^N

Let us consider an N-dimensional Euclidean space E^N. The cotangent bundle T^*E^N to this space is an $2N$-dimensional manifold naturally endowed with a symplectic structure ω. Let us choose some Euclidean coordinate system $(x^1, \dots, x^N)$ on E^N. We can extend this coordinate system to a canonical (Darboux) coordinate system $(x^1, \dots, x^N, p_1, \dots, p_N)$ on T^*E^N, which we will call an