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Integral quantizations with two basic examples

H. Bergeron [a], J.P. Gazeau [b,c,*]

[a] Univ Paris-Sud, ISMO, UMR 8214, 91405 Orsay, France

[b] Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290-180 - Rio de Janeiro, RJ, Brazil

[c] APC, UMR 7164, Univ Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France

HIGHLIGHTS

- Original approach to quantization based on (positive) operator-valued measures.
- Includes Berezin–Klauder–Toeplitz and Weyl–Wigner quantizations.
- Infinitely many such quantizations produce canonical commutation rule.
- Set of objects to be quantized is enlarged in order to include singular functions or distributions.
- Are given illuminating examples like quantum angle and affine or wavelet quantization.

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ABSTRACT

The paper concerns integral quantization, a procedure based on operator-valued measure and resolution of the identity. We insist on covariance properties in the important case where group representation theory is involved. We also insist on the inherent probabilistic aspects of this classical–quantum map. The approach includes and generalizes coherent state quantization. Two applications based on group representation are carried out. The first one concerns the Weyl–Heisenberg group and the euclidean plane viewed as the corresponding phase space. We show that a world of quantizations exist, which yield the canonical commutation rule and the usual quantum spectrum of the harmonic oscillator. The second one concerns the affine group of the real line and gives rise to an interesting regularization of the dilation origin in the half-plane viewed as the corresponding phase space.

* Correspondence to: APC, UMR 7164, Univ Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France. Tel.: +33 157276049, Mob.: +33 608829905; fax: +33 157276071.

E-mail addresses: herve.bergeron@u-psud.fr (H. Bergeron), gazeau@apc.univ-paris7.fr, jeanpierregazeau@mac.com (J.P. Gazeau).

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1. Introduction

We present in this paper an approach to quantization based on operator-valued measures, comprehended under the generic name of *integral quantization*. We particularly insist on the probabilistic aspects appearing at each stage of our procedure. The so-called Berezin or Klauder or yet Toeplitz quantization, and more generally coherent state quantization are particular (and mostly manageable) cases of this approach.

The integral quantizations include of course the ones based on the Weyl–Heisenberg group (WH), like Weyl–Wigner and (standard) coherent states quantizations. It is well established that the WH group underlies the canonical commutation rule, a paradigm of quantum physics. Actually, we show that there is a world of quantizations that follow this rule. In addition, we enlarge the set of objects to be quantized in order to include singular functions or distributions.

Our approach also includes a less familiar quantization based on the affine group of the real line. This example is illuminating and quite promising in view of applications in various domains of physics where it is necessary to take into account an impenetrable barrier.

In Section 2 we give a short overview of what we mean by quantization after recalling the basic method in use in Physics. The definition of integral quantization is proposed in Section 3. Key examples issued from group representation theory give rise to what we name *covariant integral quantizations*. We then apply the scheme to the Weyl–Heisenberg group in Section 4. As stressed on in the above, the freedom allowed by our approach gives rise to a wide range of quantum descriptions of the euclidean plane viewed as a phase space, all equivalent in the sense that they yield the canonical commutation rule. Moreover, as explained in Section 5, our approach offers the possibility of dealing with singular functions and distributions and providing in a simple way their respective quantum counterparts. Besides the euclidean plane, the half plane can be also viewed as a phase space, and Section 6 is devoted to the construction(s) of its quantum version through its unitary irreducible representation intensively used in wavelet analysis. In Section 7 we conclude our presentation with an agenda of future developments envisioned within the framework presented in this paper. In Appendix A is given a list of useful formulas concerning the Weyl–Heisenberg machinery. In Appendix B we give a compendium of previous works by Klauder where the affine group and related coherent states are also used for quantization of the half-plane, although with a different approach.

2. What is quantization?

In digital signal processing, one generally understands by *quantization* the process of mapping a large set of input values to a smaller set—such as rounding values to some unit of precision. A device or algorithmic function that performs quantization is called a quantizer. In physics or mathematics, the term has a different, and perplexing, meaning. For instance, one can find in *Wikipedia*:

> *Quantization is the process of explaining a classical understanding of physical phenomena in terms of a newer understanding known as "quantum mechanics". It is also a procedure for constructing a quantum field theory starting from a classical field theory.*

or in [1],

> *Quantization can be any procedure that associates a quantum mechanical observable to a given classical dynamical variable.*

The standard (canonical) construction is well known. It is based on the replacement, in the expression of classical observables $f(q,p)$, of the conjugate variables (q,p) (i.e., obeying $\{q,p\} = 1$) by their respective self-adjoint operator counterparts (Q,P). The latter obey the canonical commutation rule $[Q,P] = i\hbar I$. The substitution has to be followed by a symmetrization. This last step is avoided by the integral version of this method, namely the Weyl–Wigner or "phase-space" quantization [2–6] (see also [7] and references therein). The canonical procedure is universally accepted in view of its numerous experimental validations, one of the most famous and simplest one going back to the early period of Quantum Mechanics with the quantitative prediction of the isotopic effect in