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Factorization method and new potentials from the inverted oscillator

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HIGHLIGHTS

- We apply supersymmetric quantum mechanics to the inverted oscillator potential.
- The complex second-order transformations allow us to build new non-singular potentials.
- The algebraic structure of the initial and final potentials is analyzed.
- The initial potential is described by a complex-deformed Heisenberg–Weyl algebra.
- The final potentials are described by polynomial Heisenberg algebras.

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ABSTRACT

In this article we will apply the first- and second-order supersymmetric quantum mechanics to obtain new exactly-solvable real potentials departing from the inverted oscillator potential. This system has some special properties; in particular, only very specific second-order transformations produce non-singular real potentials. It will be shown that these transformations turn out to be the so-called complex ones. Moreover, we will study the factorization method applied to the inverted oscillator and the algebraic structure of the new Hamiltonians.

1. Introduction

Let us consider the following Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2, \tag{1}$$

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where m has units of mass and ω of frequency. In order to simplify, we are going to use natural units, such that $\hbar = m = 1$, to obtain

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2. \tag{2}$$

Moreover, by choosing appropriately the value of ω, three essentially different cases can be obtained:

$$\omega = \begin{cases} 1 & \text{harmonic oscillator,} \\ 0 & \text{free particle,} \\ i & \text{inverted oscillator.} \end{cases} \tag{3}$$

These are three rare examples of exactly-solvable potentials in quantum mechanics. The first one, the harmonic oscillator, is a very well known system from which the technique of creation and annihilation operators and the whole formalism of the factorization method come from. The second, the free particle, has also been largely studied. This simple system allows us to work close to the limits of quantum theory, for example, with non-square-integrable wavefunctions with plenty of physical applications such as the plane waves. The third case is not so familiar: it is called either *inverted oscillator*, *repulsive oscillator*, *inverse oscillator*, or *parabolic potential barrier*. Although it started as an exercise from Landau's book [1], its physical applications have grown since the appearance of Barton's Ph.D. Thesis (published in [2]), v.g., as an instability model, as a mapping of the 2D string theory [3], or as a toy model to study early time evolution in inflationary models [4].

It is interesting to observe that both oscillator potentials, harmonic and inverted, are simultaneously produced inside an ideal Penning trap, typically used to confine charged particles [5,6]. In its standard setup, a quadrupolar electrostatic field creates a harmonic oscillator potential along the symmetry axis of the trap, inducing confinement along this direction. In addition, in the orthogonal plane a two-dimensional inverted oscillator arises, driving the particles towards the trap walls. In order to compensate for the last effect, a static homogeneous magnetic field along the symmetry axis of the trap is also applied, but for zero magnetic field the two kinds of oscillator potentials are created inside the cavity.

Mathematically, the harmonic and inverted oscillators are very much alike, and we will show that the solutions of one can be obtained almost directly from the other one; nevertheless, we should remark that physically these two systems are very different. For example the harmonic oscillator has a discrete non-degenerate equidistant energy spectrum with square-integrable eigenfunctions, while the inverted oscillator has a continuous spectrum varying from $-\infty$ to $+\infty$, which is double degenerate, and whose eigenfunctions are not square-integrable.

On the other hand, a standard technique for generating new exactly-solvable potentials departing from a given initial one is the supersymmetric quantum mechanics (SUSY QM) (for recent reviews see [7–12]). Its simplest version, which makes use of differential intertwining operators of first-order, has been employed for generating Hamiltonians whose spectra differ from the initial one in the ground state energy level. In addition, the higher-order variants, which involve differential intertwining operators of orders larger than one [13–17], allow as well the modification of one or several excited state levels.

The SUSY techniques of first- and higher-order have been successfully applied to the harmonic oscillator [18,19] and the free particle [20,21] for generating plenty of exactly-solvable potentials. However, as far as we know, neither the first- nor the higher-order SUSY QM have been employed taking as a point of departure the inverted oscillator. In this paper we aim to fill the gap by applying the supersymmetric transformations to the inverted oscillator. In order to do that, in Section 2 we will get the general solution of the stationary Schrödinger equation (SSE) for the Hamiltonian (2) with an arbitrary energy E, which will remain valid even for $E \in \mathbb{C}$. In addition, the solutions which have a physical interpretation for the inverted oscillator will be identified. In Section 3 we are going to explore the factorization method for both systems, obtaining the bound states for the harmonic oscillator and also several sets of mathematical polynomial solutions, a class of solutions which have been of interest along the time (see e.g. [22–24]). In Section 4 we will work out the first-order SUSY QM for the inverted oscillator, while in Section 5 we will apply the second-order one in three different situations: real, confluent and complex cases. The last one will be the most important case of the paper,