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# Exact two-point resistance, and the simple random walk on the complete graph minus $N$ edges

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## ABSTRACT

An analytical approach is developed to obtain the exact expressions for the two-point resistance and the total effective resistance of the complete graph minus  $N$  edges of the opposite vertices. These expressions are written in terms of certain numbers that we introduce, which we call the Bejaia and the Pisa numbers; these numbers are the natural generalizations of the bisected Fibonacci and Lucas numbers. The correspondence between random walks and the resistor networks is then used to obtain the exact expressions for the first passage and mean first passage times on this graph.

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## 1. Introduction

A random walk on an undirected connected graph  $G$  is a process that starts at some vertex of  $G$ , and at each time step moves to one of the neighbors of the current vertex, each of them chosen with equal probability. The basic quantity relevant to random walks is the first passage time (FPT), or the hitting time, this is the expected time to hit a target node for the first time for a walker starting from a source node. This quantity is an indicator that characterizes the transport efficiency and carries much information of random walks. It has been shown that the escape probability, the FPT and the commute time (the random round trip between two nodes) of random walks are related to the effective resistance [1–3]. Therefore, the effective resistance provides an alternative way to compute the FPT. A nice interpretation of the two-point resistance  $R_{ij}$  between nodes  $i$  and  $j$  was given by Klein and Randic [4], as a measure of how close these nodes are: for unit conductances,  $R_{ij}$  is small when there are many paths between the nodes  $i$  and  $j$ , and large when there are few paths between the nodes  $i$  and  $j$ . With this interpretation in mind, the two-point resistance is sometimes called the resistance distance between nodes  $i$  and  $j$ , i.e., the two-point resistance enjoys the properties of a

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distance function. An interesting quantity related to the two-point resistance in a resistor network (undirected graph  $G = (V, E)$  with vertex set  $V$  and edge set  $E$ , with unit resistors as edges) is the total resistance distances of the graph  $G$ , denoted by  $R(G)$ . Recently, this quantity was shown to be equal to the network criticality [5]: a measure for robustness of a network to changes in traffic, topology, and community of interest. The computation of the two-point resistance of graphs is usually difficult to obtain in a closed form; however, for certain graphs with symmetries like the undirected circulant graphs, this may be possible. The undirected circulant graph [6] is a graph whose vertices can be ordered so that the adjacency matrix is a symmetric circulant matrix; the  $N$ -cycle and the complete graphs are examples of circulant graphs. Chau and Basu [7] recently derived a formula to compute the FPT of the random walk on the  $N$ -cycle graph with  $2p$  neighbors, i.e., the undirected circulant graph of the type  $C_N(1, 2, \dots, p)$ . Their formula is based on Lovasz's formula for the expected hitting time of a random walk on a finite graph [8]. Wu in his paper [9] on the theory of resistor networks, derived a formula to compute the two-point resistance between any two nodes in terms of the eigenvalues and the eigenvectors of the Laplacian matrix associated with the finite electrical network. Using this formula, he obtained the two-point resistance of the complete and the cycle graphs. By diagonalizing the Laplacian matrix associated with the  $N$ -cycle graph with  $2p$  nearest neighbors and using Wu's formula we obtain a formula to compute the two-point resistance between any two vertices of this graph. Then it is not difficult to show that this formula, when multiplied by the number of edges  $|E|$ , is identical to the FPT given in [7]. This is expected, since we are dealing with undirected circulant graphs that enjoy rotational symmetry, each vertex of these graphs is a vertex-transitive, i.e., looks the same from any vertex, then the first passage time is symmetric under the exchange of the vertices. Therefore, we may as well consider that the random walk has started at vertex 0, and after some steps reaches a given vertex, say  $l$ . Using the commute time formula given by Chandra et al. [3],  $C_{ij} = 2|E|R_{ij}$ , then the first passage time  $H_{0l}$  may be written as  $H_{0,l} = |E|R_{0,l}$ . For example, the two-point resistance between the vertex 0 and any other vertex  $l$  of the  $N$ -cycle is  $R_{0,l} = l(1 - l/N)$ , and since the number of edges is  $|E| = N$ , then the expression for the FPT of the random walk on the  $N$ -cycle gives  $H_{0,l} = l(N - l)$ . This result was derived previously using probabilistic techniques on graphs [10]. In this paper, we give the exact expression for the two-point resistance between any two vertices of the complete graph minus  $N$  edges of the opposite vertices. If  $N$  is odd, this graph is denoted by  $K_N^{-N}$ . If  $N$  is even, then the complete graph minus  $N/2$  edges of the opposite vertices is known as the cocktail-party graph [6], and in this case the two-point resistance computations are straightforward, unlike in this paper. The general formula to compute the two-point resistance of the graph  $K_N^{-N}$  turns out to be given by trigonometrical power sums. To obtain the exact two-point resistance, extra care is needed when using a formula by Schwatt [11] on trigonometrical power sums, since the latter does not give the right answer when the powers are congruent to  $N$ . Therefore, we have to solve this problem first before doing our computations. As a consequence, computing the two-point resistance is not direct and is done in steps; once the right formula for the trigonometrical power sum is obtained, we use the binomial coefficients representation by residues, and the linearity property of the residue operator. This property plays an important role in our paper, enabling us to avoid carrying out certain sums of binomials, such as  $\sum_{p=1}^{\lfloor j/N \rfloor} (-1)^{pN} \binom{2j}{j-pN}$ , which turns out to be an open mathematical problem in combinatorics,<sup>1</sup> with the only known closed formula for this sum being for  $N = 1, 2, 3$ . Then, using the Chebyshev polynomial of the first kind, and introducing certain numbers, which we call the Bejaia and the Pisa numbers, the two-point resistance is obtained. We find that the names Bejaia and Pisa fit nicely here, simply because Fibonacci started thinking about his famous numbers while he was in Bejaia and wrote them when he went back to Pisa. These numbers have nice properties like the Fibonacci and the Lucas numbers. More precisely, these numbers are the natural generalizations of the bisected Fibonacci and Lucas numbers, that is,  $F_{2n} = \frac{1}{\sqrt{5}} \left( \frac{3+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{3-\sqrt{5}}{2} \right)^n$ , and  $L_{2n} = \left( \frac{3+\sqrt{5}}{2} \right)^n + \left( \frac{3-\sqrt{5}}{2} \right)^n$  respectively. The total effective resistance, and important parameters

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