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Bloch space structure, the qutrit wavefunction and atom–field entanglement in three-level systems

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ABSTRACT

We have given a novel formulation of the exact solutions for the lambda, vee and cascade three-level systems where the Hamiltonian of each configuration is expressed in the $SU(3)$ basis. The solutions are discussed from the perspective of the Bloch equation and the atom–field entanglement scenario. For the semiclassical systems, the Bloch space structure of each configuration is studied by solving the corresponding Bloch equation and it is shown that at resonance, the eight-dimensional Bloch sphere is broken up into two distinct subspaces due to the existence of a pair of quadratic constants. Because of the different structure of the Hamiltonian in the $SU(3)$ basis, the non-linear constants are found to be distinct for different configurations. We propose a possible representation of the qutrit wavefunction and show its equivalence with the three-level system. Taking the bichromatic cavity modes to be in the coherent state, the amplitudes of all three quantized systems are calculated by developing an Euler angle based dressed state scheme. Finally following the Phoenix–Knight formalism, the interrelation between the atom–field entanglement and population inversion for all configurations is studied and the existence of collapses and revivals of two different types is pointed out for the equidistant cascade system in particular.

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1. Introduction

In the atom–field interaction scenario, the level structure of an atom leads to the prediction of a wide range of experimentally verifiable coherent phenomena. Probably the most notable among them is the observation of the collapse and revival of the Rabi oscillation [1] which unequivocally proves the granular structure of the photon. This phenomenon is indeed a prediction of the Jaynes–Cummings model—an idealized two-level system consists of an atom with two distinct quantized levels interacting with a monochromatic quantized cavity field [2,3]. An immediate extension of the two-level system is the three-level system which is generally classified into lambda, vee and cascade configurations, respectively. Such configurations are in the purview of present studies because they exhibit a rich class of coherent phenomena such as two-photon coherence [4], the double-resonance process [5], three-level super-radiance [6], resonance Raman scattering [7], population trapping [8], trilevel echoes [9], STIRAP [10], quantum jumps [11], the quantum Zeno effect [12], electromagnetically induced transparency [13] etc. From these studies it is quite transparently obvious that increase of the number of levels not only can generate a large number of quantum-optical effects, but also enables us to develop a suitable control mechanism which is extremely important from the experimental point of view. Thus the three-level configuration, the simplest representative of the multi-level system, demands careful inspection from time to time in its own right.

It is well known that the Hamiltonians of the lambda, vee and cascade three-level system types can be described using the atomic basis operator, $\hat{\sigma}_{\mu\nu} \equiv |\mu\rangle\langle\nu|$ $(\mu, \nu = 1, 2, 3)$, where the solution is carried out with two-photon resonance and equal detuning conditions as supplementary conditions [14,15]. Apart from this treatment, another equivalent way of dealing with the three-level system is by the Bloch equation technique, where the eight Bloch vectors are defined on the eight-dimensional Bloch sphere S^7 [16–18]. This method was first initiated by Eberly and Hioe who pointed out the relationship of the three-level system with the $SU(3)$ group [16]. Their investigation revealed that the quadratic Casimir of the $SU(3)$ group is manifested through the existence of some non-linear constants, which gives rise to a non-trivial structure of the Bloch space of such a system [18,19]. Later, this result was obtained by solving the pseudo-spin equation [20] and also by the Floquet theory technique [21]. However, in the Bloch equation approach, the lambda, vee and cascade three-level systems are found to be generated by changing the position of the intermediate level E_2, i.e., the energy levels are arranged as $E_2 > E_3 > E_1$, $E_1 > E_3 > E_2$ and $E_3 > E_2 > E_1$, as shown in Fig. 1 of Ref. [18]. In consequence, irrespectively of the configuration, the interaction term for any one of these three-level systems in the atomic operator basis is given by $H_I^A = g_1|1\rangle\langle 2| + g_2|2\rangle\langle 3| + h.c.$ $(A = \Lambda, V$ and $\Xi)$. The pitfall of having identical structures of the Hamiltonians for different configurations is that this leads to same set of non-linear constants, which is undesirable because the three-level systems are intrinsically different from one another.

Apart from that, there is another reason for studying the Bloch space structure of three-level systems. In quantum information theory parlance, the qubit is "designated" by various points on the aforesaid unit Bloch sphere [22,23]. A natural extension of the qubit is the qutrit system, which is generally expressed in the computational basis: $|-\rangle$, $|0\rangle$ and $|+\rangle$; it has drawn wide attention in recent years [24–26]. Although there exist several suggestions for implementing the qutrit system either by treating it as the transverse spatial modes of single photons [27], or through the polarization states of the biphoton field [28,29], significant progress has been made mainly by identifying the qutrit with the three-level system driven by bichromatic laser fields [30]. However, in spite of the significant progress, a proper definition of the qutrit wavefunction and its explicit relation with the Bloch space structure of the three-level system are not available.

Recently a number of quantum-optical systems have come under intense scrutiny with a view to the experimental implementation of various quantum information protocols where the concurrence is considered the most useful dynamical parameter of evolution [31]. The primary reason for such a study is that it is capable of deciphering the nature of the entanglement between two non-local objects which is the key resource in communicating information. For example, in the phenomenon of *entanglement sudden death*, the entanglement between two non-local two-level systems is studied by considering the time evolution of the concurrence [32]. Since the two-level system is essentially an atom–field composite system, it is equally important to understand the dynamics of the entanglement