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Spectral parameter power series representation for Hill's discriminant

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ABSTRACT

We establish a series representation of the Hill discriminant based on the spectral parameter power series (SPPS) recently introduced by Kravchenko. We also show the invariance of the Hill discriminant under a Darboux transformation and employing the Mathieu case the feasibility of this type of series for numerical calculations of the eigenspectrum.

1. Introduction

There is recent strong interest in differential equations with periodic coefficients due to their numerous applications in modern material sciences and engineering. The main goal in their theoretical framework is the description of the spectrum and getting the periodic and quasiperiodic solutions. In the case of linear second-order ordinary differential equation, a function of the spectral parameter known as Hill's discriminant is the basic quantity containing important information both about the spectrum of the differential operator and also about the construction of the (quasi) periodic solutions.

This paper focuses on a representation of the Hill discriminant which is given in the form of a power series in the spectral parameter. This representation is obtained by using recent results of Kravchenko [1–3]. It is worth mentioning that since the Hill discriminant for each value of the spectral parameter can be obtained from a couple of corresponding linearly independent solutions, any available representation for these solutions leads to a representation of Hill's discriminant. Nevertheless, in general no easy and practically treatable representation for Hill's discriminant is known unless for the case of some well-studied equations such as the Mathieu and the Lamé equations.

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Long ago, Jagerman [4] introduced and studied in detail the so-called cardinal series representation of Hill's discriminant. In the famous book of Magnus and Winkler [5] the Hill discriminant for Schroedinger type equations is expressed as an infinite determinant involving the Fourier coefficients of the potential as well as the spectral parameter. The phase-integral method is used by Fröman [6] to obtain a representation involving a matrix whose entries are complicated phase integrals, while Boumenir [7] wrote it in terms of integrals derived from the inverse spectral theory.

In all the aforementioned series representations the spectral parameter enters in a quite sophisticated way with all the terms being functions of the spectral parameter. Instead, our result herein gives the Hill discriminant in the form of a power series in the spectral parameter with the series coefficients independent of it and calculated only once, i.e., for a single value of the spectral parameter. Moreover, its practical implementation is easy and as an illustration we show in the present work the numerical results obtained for the Mathieu equation.

We also show that the SPPS representation gives additional information not only concerning the original equation but also on its Darboux-related partners. As a corollary, we prove the invariance of the Hill discriminant under the Darboux transformation under some additional conditions.

2. Hill's type equations

The Sturm–Liouville differential equation

$$L[f(x,\lambda)] = -(p(x)f'(x,\lambda))' + q(x)f(x,\lambda) = \lambda f(x,\lambda), \tag{1}$$

with T-periodic coefficients $p(x)$ and $q(x)$ and real parameter λ is known as of Hill type. We first recall some necessary definitions and basic properties associated with the Eq. (1) from the Floquet (Bloch) theory. For more details see, e.g., [5,8]. In what follows we assume that $p(x) > 0$, $p'(x)$ and $q(x)$ are continuous bounded functions.

For each λ there exists a fundamental system of solutions, i.e., two linearly independent solutions of (1) $f_1(x,\lambda)$ and $f_2(x,\lambda)$ which satisfy the initial conditions

$$f_1(0,\lambda) = 1, \quad f_1'(0,\lambda) = 0, \quad f_2(0,\lambda) = 0, \quad f_2'(0,\lambda) = 1. \tag{2}$$

Then the Hill discriminant associated with Eq. (1) is defined as a function of λ as follows

$$D(\lambda) = f_1(T,\lambda) + f_2'(T,\lambda).$$

Employing $D(\lambda)$ one can easily describe the spectrum of the corresponding equation. Namely, the values of λ for which $|D(\lambda)| \leqslant 2$ form the allowed bands or stability intervals meanwhile the values of λ such that $|D(\lambda)| > 2$ belong to forbidden bands or instability intervals [5]. The band edges (values of λ such that $|D(\lambda)| = 2$) represent the discrete spectrum of the operator, i.e., they are the eigenvalues of the operator with periodic ($D(\lambda) = 2$) or antiperiodic ($D(\lambda) = -2$) boundary conditions. The eigenvalues λ_n, $n = 0,1,2,\ldots$ form an infinite sequence $\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3\ldots$, and an important property of the minimal eigenvalue λ_0 is the existence of a corresponding periodic nodeless solution $f_0(x,\lambda_0)$ [5]. In general solutions of (1) are not of course periodic, and one of the important tasks related to Sturm–Liouville equations with periodic coefficients is the construction of quasiperiodic solutions. In this paper, we use the matching procedure from [9] for which the main ingredient is the pair of solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ of (1) satisfying conditions (2). Namely, using $f_1(x,\lambda)$ and $f_2(x,\lambda)$ one obtains the quasiperiodic solutions $f_\pm(x+T) = \beta_\pm f_\pm(x)$ as follows

$$f_\pm(x,\lambda) = \beta_\pm^n F_\pm(x-nT,\lambda), \quad \begin{cases} nT \leqslant x < (n+1)T \\ n = 0, \pm 1, \pm 2, \ldots \end{cases}, \tag{3}$$

where $F_\pm(x,\lambda)$ are the so-called self-matching solutions, which are the following linear combinations $F_\pm(x,\lambda) = f_1(x,\lambda) + \alpha_\pm f_2(x,\lambda)$ with $\alpha_\pm$ being roots of the algebraic equation $f_2(T,\lambda)\alpha^2 + (f_1(T,\lambda) - f_2'(T,\lambda))\alpha - f_1'(T,\lambda) = 0$. The Bloch factors $\beta_\pm$ are a measure of the rate of increase (or decrease) in magnitude of the self-matching solutions $F_\pm(x,\lambda)$ when one goes from the left end of the cell to the right end, i.e., $\beta_\pm(\lambda) = \frac{F_\pm(T,\lambda)}{F_\pm(0,\lambda)}$. The values of $\beta_\pm$ are directly related to the Hill discriminant, $\beta_\pm(\lambda) = \frac{1}{2}(D(\lambda) \mp \sqrt{D^2(\lambda) - 4})$, and obviously at the band edges $\beta_+ = \beta_- = \pm 1$ for $D(\lambda) = \pm 2$, correspondingly.