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The effect of singular potentials on the harmonic oscillator

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ABSTRACT

We address the problem of a quantum particle moving under interactions presenting singularities. The self-adjoint extension approach is used to guarantee that the Hamiltonian is self-adjoint and to fix the choice of boundary conditions. We specifically look at the harmonic oscillator added of either a δ-function potential or a Coulomb potential (which is singular at the origin). The results are applied to Landau levels in the presence of a topological defect, the Calogero model and to the quantum motion on the noncommutative plane.

1. Introduction

The harmonic oscillator models a very large number of physical systems. It appears as the interaction between atoms in the elastic crystal, as the effective potential acting on electrons moving in a uniform magnetic field (Landau levels), in the quantization of fields, etc. Localized interactions are less ubiquitous but nonetheless important. They appear as singularities, like a δ-function, for instance, which can be very handy when modelling very short-ranged interactions [1]. These are the so-called contact interactions, which appear in such diversified physical systems as nanoscale quantum devices [2] and in the optics of thin dielectric layers [3], for example. In quantum mechanics, singularities and pathological potentials, in general, are often dealt with by some kind of regularization. A common approach to ensure that the wave function in the presence of a singularity is square-integrable (and therefore might be associated to a bound state) is to force it to vanish on the singularity. More appropriately, an analysis based on the self-adjoint extension method [4], broadens the boundary condition

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possibilities that still give bound states. The physics of the problem determines which of these possibilities is the right one, leaving no ambiguities, like done in Refs. [5,6] for a cosmic string.

In a recent article [7], two of us used self-adjoint extension to study the quantum dynamics of a free particle on a conical surface as a toy model. This was motivated by a previous work [8] where we studied a gravitational analogue of the bound-state Aharonov–Bohm effect. Instead of a magnetic flux we considered a curvature flux provided by a cosmic string. This object, geometrically, corresponds to a Minkowiski space–time with a conical singularity, that is, a line element given by

$$ds^2 = c^2dt^2 - dz^2 - d\rho^2 - \alpha^2\rho^2 d\theta^2, \tag{1}$$

where α is related to the linear mass density of the string. For this choice of coordinates the conical singularity lies on the z-axis. Notice that the ordinary cone has its geometry described by the $t = const.$, $z = const.$ section of the cosmic string space–time. In [8] we solved the Schrödinger equation for a free particle in the background given by (1) and found a bound state without the need of confinement, a requirement for the usual bound-state Aharonov–Bohm effect to appear [9]. In [7] we studied the case of a particle moving on a cone under the influence of a pathological potential which goes with $1/\rho^2$. The curvature singularity of the cone enters the Schrödinger equation as the geometric potential [10]

$$U_{geo} = -\frac{\hbar^2}{2M}(H^2 - K), \tag{2}$$

where H and K are, respectively, the mean and the Gaussian curvature of the surface. This is necessary due to the embedding of the surface in three-dimensional space [10]. It is the Gaussian curvature the one that contributes with the δ-function [7]:

$$K = \left(\frac{1-\alpha}{\alpha}\right)\frac{\delta(\rho)}{\rho}. \tag{3}$$

The mean curvature leads to the pathological potential since [7]

$$H^2 = \frac{1-\alpha^2}{4\alpha^2\rho^2}. \tag{4}$$

In this work we address the problem of a harmonic oscillator on a plane with a single δ-function singularity located at the origin of a polar coordinate system. Besides the obvious connection with references [8,7], this problem is related to applications like the study of Landau levels in a medium with a topological defect, the Calogero model, and the dynamics of a charged particle on the noncommutative plane in the presence of a flux tube, all of which we discuss here. It is important to notice that the influence of topology on Landau levels has been addressed over the years in the context of spaces with topological defects [11,12]. The difference between this work and the previous ones is that we are including the coupling between the singularity and the eigenfunctions, that is, we are including a coupling between the eigenfunctions and a short-ranged potential (modeled by a δ-function interaction). We deal with this problem via the self-adjoint extension approach [4].

2. Two-dimensional harmonic oscillator in the presence of a δ-function singularity

The Hamiltonian of the harmonic oscillator in two-dimensional space is given by

$$H_o = \frac{p_x^2}{2M} + \frac{p_y^2}{2M} + \frac{\omega_x^2 x^2}{4} + \frac{\omega_y^2 y^2}{4}, \tag{5}$$

where the factor $\frac{1}{4}$ is written for convenience. In polar coordinates (r, φ) it can be written as

$$H_o = -\frac{\hbar^2}{2M}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right] + \frac{\omega^2 r^2}{4}, \tag{6}$$