Annals of Physics 373 (2016) 10–27

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

# Cyclic groups and quantum logic gates

Arash Pourkia [a], J. Batle [b,*], C.H. Raymond Ooi [c]

[a] *Mathematics division, College of Engineering, American University of the Middle East, 220 Dasman, 15453, Kuwait*

[b] *Departament de Física, Universitat de les Illes Balears, 07122 Palma de Mallorca, Balearic Islands, Spain*

[c] *Department of Physics, University of Malaya, 50603 Kuala Lumpur, Malaysia*

ARTICLE INFO

*Article history:*
Received 21 March 2016
Accepted 28 June 2016
Available online 5 July 2016

*Keywords:*
Quantum computation
Yang–Baxter equation
Quantum gates
Cyclic groups
Berry phase

ABSTRACT

We present a formula for an infinite number of universal quantum logic gates, which are 4 by 4 unitary solutions to the Yang–Baxter (Y–B) equation. We obtain this family from a certain representation of the cyclic group of order $n$. We then show that this *discrete* family, parametrized by integers $n$, is in fact, a small sub-class of a larger *continuous* family, parametrized by real numbers $\theta$, of universal quantum gates. We discuss the corresponding Yang-Baxterization and related symmetries in the concomitant Hamiltonian.



## 1. Introduction

Quantum correlations lie at the heart of quantum information theory and quantum computation. They are responsible for some tasks that possess no classical counterpart. Among those correlations, entanglement is perhaps one of the most fundamental and non-classical feature exhibited by quantum systems [1–10].

In recent years, a different approach to quantum entanglement has been developed, with the ultimate goal of achieving fault-tolerant quantum computation. In particular, the works of Kauffman and Lomonaco [11–13], on the connections between quantum entanglement, topological entanglement, and quantum computing, have brought the unitary solutions to the Yang–Baxter (Y–B) equation, i.e. Eq. (1.1), to the center of attention.

Let $V$ be a $n$ dimensional Vector (Hilbert) space over a field $F$ (for us $F = \mathbb{C}$ the field of complex numbers), and let $R : V \otimes V \rightarrow V \otimes V$ be a linear map. When $R$ is unitary (i.e. $R^{-1} = R^{\dagger}$ the

* Corresponding author.
*E-mail addresses:* apourki@uwo.ca (A. Pourkia), jbv276@uib.es (J. Batle).

http://dx.doi.org/10.1016/j.aop.2016.06.023

conjugate transpose of $R$), it could be considered as a quantum logic gate, in quantum computing. In the study of quantum entanglement in quantum computing, it is critically important when $R$ is entangling i.e. when it creates entangled states from non-entangled ones [11–13].

A linear map $R : V \otimes V \to V \otimes V$ is said to be a solution to parameter-independent Y–B equation, if it satisfies the relation, [14,15].

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R), \tag{1.1}$$

where $I$ is the identity map on $V$. Since $R$ is a linear map one can present it as a $n^2$ by $n^2$ matrix for some basis of $V$. If $R$ is invertible, it provides an infinite family of braid group representations [14,15], which, in turn, yields some invariants of links [16–18]. This, in a sense, is where the topological entanglement being studied [11–13].

The so called unitary braiding operators, i.e. unitary solutions to Y–B equations, and the relations between quantum and topological entanglement in quantum computing, have been studied extensively by many authors, in the last decade or so. For example some of the works that we will refer to in this paper are in [19–26].

On the other hand, it is a very well known fact, [14,15] that a natural source of solutions to Y–B equation is from quasitriangular Hopf algebras. In a quasitriangular Hopf algebra, $(H, R)$, $H$ is a Hopf algebra and $R = \sum_i \mathfrak{r}_i \otimes \mathfrak{s}_i$ (usually summation understood and eliminated) is an invertible element in $H \otimes H$ satisfying certain relations [14,15]. This element $R$ satisfies the following version of parameter-independent Y–B equation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.2}$$

where $R_{12} = \sum_i \mathfrak{r}_i \otimes \mathfrak{s}_i \otimes 1$, $R_{13} = \sum_i \mathfrak{r}_i \otimes 1 \otimes \mathfrak{s}_i$, and $R_{23} = \sum_i 1 \otimes \mathfrak{r}_i \otimes \mathfrak{s}_i$.

This property of $R$ implies that $\tau R$ gives rise to representations of *Artin braid group* $B_n$ or correspondingly to a solution to Y–B equation (1.1). Here $\tau$ is the flip (swap) map given by, $\tau(x \otimes y) = y \otimes x$.

In this paper we focus on two dimensional vector spaces $V$ over, $\mathbb{C}$, the field of complex numbers. In this case, $R : V \otimes V \to V \otimes V$ can be represented by a 4 by 4 matrix with entries in $\mathbb{C}$, with respect to a basis of $V$.

Usually, the preferred basis for two qubit states or gates is the so called computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In our case, it will prove at some point convenient to employ the so called Bell basis of maximally correlated states, which are of the form

$$|\Phi^{\pm}\rangle = \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}}, \qquad |\Psi^{\pm}\rangle = \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}}. \tag{1.3}$$

The employment of a unitary solution to the Y–B equation will eventually bring us to a description of Hamiltonians in terms of the computational basis, which shall be represented in the form of tensor products of the generators of the $su(2)$-group, that is, the familiar Pauli matrices.

This paper is organized as follows. In Section 2, we obtain quantum logic gates from cyclic groups, $C_n$, for any order $n$, via the quasitriangular structure on their group Hopf algebra. The corresponding proof is given in the Appendix. The corresponding Yang-Baxterization is performed in Section 3. The analysis of the ensuing family of Hamiltonians and the concomitant physical applications appears is Section 4. General continuous quantum gates are studied in Section 5. A second approach to quantum logic gates from cyclic groups is given in Section 6. Finally, some conclusions are drawn in Section 7.

## 2. Quantum gates correspond to cyclic groups

In this section we state one of our main lemmas, whose outcome helps us to obtain an infinite family of 4 by 4 unitary solutions to Yang–Baxter (Y–B) equation (1.1), from the cyclic group of order $n$. The matrices $\mathfrak{B}_n$ in this family are entangling universal logic gates, for $n \neq 2, 4$. We then prove that a deformation of $\mathfrak{B}_n$ by a phase factor has all the above mentioned properties as well.

We recall [14,15], a natural source of solutions to the (parameter-independent) Y–B equation (1.1), and correspondingly a source for representations of Artin braid group $B_n$ (do not be confused with