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Energy levels of one-dimensional systems satisfying the minimal length uncertainty relation

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ABSTRACT

The standard approach to calculating the energy levels for quantum systems satisfying the minimal length uncertainty relation is to solve an eigenvalue problem involving a fourth- or higher-order differential equation in quasiposition space. It is shown that the problem can be reformulated so that the energy levels of these systems can be obtained by solving only a second-order quasiposition eigenvalue equation. Through this formulation the energy levels are calculated for the following potentials: particle in a box, harmonic oscillator, Pöschl–Teller well, Gaussian well, and double-Gaussian well. For the particle in a box, the second-order quasiposition eigenvalue equation is a second-order differential equation with constant coefficients. For the harmonic oscillator, Pöschl–Teller well, Gaussian well, and double-Gaussian well, a method that involves using Wronskians has been used to solve the second-order quasiposition eigenvalue equation. It is observed for all of these quantum systems that the introduction of a nonzero minimal length uncertainty induces a positive shift in the energy levels. It is shown that the calculation of energy levels in systems satisfying the minimal length uncertainty relation is not limited to a small number of problems like particle in a box and the harmonic oscillator but can be extended to a wider class of problems involving potentials such as the Pöschl–Teller and Gaussian wells.

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1. Introduction

The existence of a minimal length uncertainty which is predicted in modern theories of gravity [1–6] and relativity [7–9] cannot be accounted for by the Heisenberg uncertainty principle (HUP). It is known that larger energies are needed to probe shorter distances in accordance with the HUP. But when the scattering energy is comparable to the Planck energy scale, the gravitational effects become dominant and the foamy structure of spacetime manifests itself through the generalized uncertainty principle (GUP) [6]. This is consistent with high energy scattering of strings [10–13] and gedanken experiments [4]. The minimal length uncertainty relation (Eq. (5)) is one of the GUPs discussed in the literature and is the one considered in this paper [14,15]. Other GUPs include not only a minimal uncertainty in position but also maximal uncertainty in momentum or minimal uncertainty in momentum [16,17]. Two recent reviews [6,18] can be consulted for the various kinds of GUP.

The minimal length uncertainty relation overcomes the shortcomings of HUP at high energies but with the consequence, known as the universality of quantum gravity corrections, of modifying all Hamiltonians [14,19,20]. Consequences of the minimal length uncertainty relation include the positive shift in the energy levels of bound states [14,15,21–25] and the resonant energies of scattering states [26–28] and the reduction of the degrees of freedom of a statistical system at high temperatures [29–32]. It has also been suggested that theories which are consistent with the minimal length uncertainty relation can be used to describe the nonpointlike nature of particles [21,33–35].

The standard approach in calculating the energy levels for systems satisfying the minimal length uncertainty relation is to solve an eigenvalue equation which is a fourth- or higher-order differential equation in quasiposition space (Eq. (1)) [14,15,20,21,24,26,28,36,37]. The energy levels have also been obtained by solving the eigenvalue problem in momentum space for the following problems: harmonic oscillator, linear potential, Coulomb potential, Kratzer potential, Dirac delta potential, and double Dirac delta potential [14,15,25,38–40]. The momentum space approach is useful only when the transformed differential equation is simpler to solve compared to the original quasiposition space differential equation. For example, the momentum space approach to the problem of a particle in a harmonic oscillator potential leads to a second-order differential equation but for a particle in a Gaussian potential well the same technique leads to an infinite-order differential equation. To calculate the energy levels for a wider class of potentials including the Pöschl–Teller and Gaussian potential wells a new technique has to be developed. This issue is addressed in this paper. By treating an arbitrary potential as the limiting case of a series of infinitesimally thin rectangular strips, the second-order quasiposition eigenvalue equation (Eq. (18)) on which the calculation of energy levels will be based is derived by compounding solutions corresponding to constant potential regions (Section 2). Through this formulation the energy levels of a particle in a box and the harmonic oscillator are compared with literature values and the energy levels of a particle in the Pöschl–Teller well, Gaussian well, and double-Gaussian well are calculated (Section 3). For the particle in a box, the second-order quasiposition eigenvalue equation is a second-order differential equation with constant coefficients. For the harmonic oscillator, Pöschl–Teller well, Gaussian well, and double-Gaussian well, a method involving the use of Wronksians has been applied to solve the second-order quasiposition eigenvalue equation. The results and advantages of the approach are summarized (Section 4). The recipe for calculating the energy levels for an arbitrary potential is also given.

2. Second-order quasiposition eigenvalue equation

In this section, the second-order quasiposition eigenvalue equation (Eq. (18)) is derived using the following approach: first, it is shown that the solutions to the Schrödinger equation (Eq. (1)) in regions of constant potential are plane waves (exponential functions) with minimal length-modified expressions for the wave number (decay constant). Then, an equation that is satisfied for an arbitrary potential (which is treated as the limiting case of a series of infinitesimally thin rectangular strips) is built using the method of transfer matrices [41,42]. In the next section it is shown that the resulting equation (Eq. (18)) which is called the second-order quasiposition eigenvalue equation can be used to calculate the energy levels of quantum systems.