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Time, classical and quantum

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ABSTRACT

We propose a new point of view regarding the problem of time in quantum mechanics, based on the idea of replacing the usual time operator **T** with a suitable real-valued function T on the space of physical states. The proper characterization of the function T relies on a particular relation with the dynamical evolution of the system rather than with the infinitesimal generator of the dynamics (Hamiltonian). We first consider the case of classical hamiltonian mechanics, where observables are functions on phase space and the tools of differential geometry can be applied. The idea is then extended to the case of the unitary evolution of pure states of finite-level quantum systems by means of the geometric formulation of quantum mechanics. It is found that T is a function on the space of pure states which is not associated with any self-adjoint operator. The link between T and the dynamical evolution is interpreted as defining a simultaneity relation for the states of the system with respect to the dynamical evolution itself. It turns out that different dynamical evolutions lead to different notions of simultaneity, i.e., the notion of simultaneity is a dynamical notion.

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1. Introduction

The problem of time in quantum mechanics is a beautiful and subtle one. We can formalize it with a simple question, namely, is there a self-adjoint operator we can associate to time in quantum mechanics? Or, even better, is time a quantum observable?

There are many experimental instances in which this question makes sense because time seems to acquire an observable character. For example, we can think of the time of arrival of a particle in a detector, the time of occurrence of a specific event, or the tunneling time of a particle under the influence of a potential barrier.

In standard quantum mechanics, observable quantities are described by means of self-adjoint linear operators on the Hilbert space of the system. In this setting, a time observable **T** would be characterized as a self-adjoint operator **T** which is canonically conjugated to the Hamiltonian operator **H** of the system:

$$[\mathbf{H}, \mathbf{T}] = -\imath\hbar\,\mathbb{I}. \tag{1}$$

In contrast with CCRs relating position and momentum, the commutation relation between **T** and **H** is plagued by severe technical difficulties. In [1], Pauli realized that a self-adjoint operator **T** canonically conjugated to the Hamiltonian operator **H** does not exist whenever the spectrum of **H** is bounded from below. Pauli's proof was not rigorous, and, to be fair, he never claimed it to be so. However, it took some time for the rigorous mathematical formulation of the problem to be settled (see [2], and [3]), and, in the meantime, different strategies to cope with the problem have been proposed. For instance, attention has been given to the possibility of relaxing the self-adjointness condition for the time observable **T**. In this direction, of particular interest is the construction of a maximally symmetric time operator **T** which is canonically conjugated to the Hamiltonian operator **H** of the 1-dimensional free particle given by Aharonov and Bohm in [4]. This operator is a sort of canonical quantization of the classical passage time of Newtonian mechanics, and thus, its physical interpretation is related to the experimental concepts of passage time, and of time of flight. Another change of perspective occurred, and efforts were, and are made to construct a positive operator-valued measure (POVM) having a particular covariance property with respect to the dynamics, and that can be reasonably interpreted as a time POVM ([5–7], and [8]). In this setting, the physical interpretation of the time POVM constructed in [9] is related to the experimental concept of time of occurrence. Finally, some interesting counterexamples to Pauli's theorem have been given. Among the most interesting ones is the case of a phase operator constructed by Galindo [10] and Garrison and Wong [11], which is a bounded, self-adjoint operator canonically conjugated to the number operator, and thus with the Hamiltonian operator, of the 1-dimensional quantum harmonic oscillator. The physical interpretation of this operator is in some sense related to the quantum-mechanical formulation of the action–angle variables exposed by Dirac in [12].

From this brief discussion we can extract two important facts. First of all, time in quantum mechanics is a dynamical quantity which is intimately connected with the specific dynamical evolution of the system and with specific experimental questions. Second, it seems that self-adjoint operators are simply not enough to handle the problem of time in quantum mechanics, and different mathematical objects may be appropriate to treat different aspects of time. In this article, we focus on the simultaneity aspect of time in quantum mechanics, and propose to describe it by means of a real-valued function T on the space of physical states, which we call a time function, satisfying a particular equivariance condition with respect to the dynamical evolution of the system.

In accordance with Einstein's theory of special relativity, we recognize two different but related aspects of our common perception of time in physical phenomena. On the one hand, time appears as an evolution parameter, a sort of ordering label by means of which we formalize the perception of the causal aspect of "before" and "after". Following an heuristic argument, the mathematical object that captures this aspect of time in a spacetime framework is a vector field, say $\frac{\partial}{\partial t}$. Given an integral curve $\gamma_m(\tau)$ of $\frac{\partial}{\partial t}$ starting at $m = \gamma_m(0)$, the parameter τ "measures" causality in the sense that $m_1 = \gamma_m(\tau_1)$ casually precedes $m_2 = \gamma_m(\tau_2)$ if and only if $\tau_1 < \tau_2$, and thus, all the events lying on $\gamma_m(\tau)$ are interpreted as causally connected through the spacetime evolution determined by $\frac{\partial}{\partial t}$. It is