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Simultaneous ordinary and type A $\mathcal{N}$-fold supersymmetries in Schrödinger, Pauli, and Dirac equations

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Abstract

We investigate physical models which possess simultaneous ordinary and type A $\mathcal{N}$-fold supersymmetries, which we call type A $(\mathcal{N}, 1)$-fold supersymmetry. Inequivalent type A $(\mathcal{N}, 1)$-fold supersymmetric models with real-valued potentials are completely classified. Among them, we find that a trigonometric Rosen–Morse type and its elliptic version are of physical interest. We investigate various aspects of these models, namely, dynamical breaking and interrelation between ordinary and $\mathcal{N}$-fold supersymmetries, shape invariance, quasi-solvability, and an associated algebra which is composed of one bosonic and four fermionic operators and dubbed type A $(\mathcal{N}, 1)$-fold superalgebra. As realistic physical applications, we demonstrate how these systems can be embedded into Pauli and Dirac equations in external electromagnetic fields.

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1. Introduction

Ever since the formulation of supersymmetric quantum mechanics [1], a much deeper understanding of structure of various solvable potentials in non-relativistic quantum mechanics has been gained through the ideas of supersymmetry (SUSY) (for a good review, see, e.g. [2,3]). For instance, seemingly distinct potentials, e.g., an infinite square-well and a cosec^2 potential, are in fact supersymmetric partners, and their respective energy levels are connected by supercharges. The methods of SUSY have also been applied to obtaining exact spectra of quantum systems with multi-component wave functions, such as Pauli and Dirac equations. It is based on the fact that, for some electromagnetic field configurations, Pauli and Dirac equations possess intrinsic SUSY structure, and different components of a wave function are related by some supercharges [4,5]. More recently, the method of SUSY has been employed to construct quasi-exactly solvable potentials in the Pauli and Dirac equations [6,7]. For quasi-exactly solvable systems, it is only possible to determine algebraically a part of the spectrum but not the whole spectrum [8–12].

Recently, the concepts of SUSY and quasi-exact solvability were combined within the framework of so-called $\mathcal{N}$-fold supersymmetry [13], which is a natural generalization [14] of ordinary supersymmetric quantum mechanics. Among various nonlinear extensions of ordinary SUSY such as parasupersymmetry [15–17], fractional supersymmetry [18], and so on, $\mathcal{N}$-fold SUSY is characterized by the fact that anti-commutators of fermionic operators are polynomials of degree (at most) $\mathcal{N}$ in bosonic operators. It has been proved in a generic way that $\mathcal{N}$-fold SUSY is essentially equivalent to (weak) quasi-solvability (this latter term, in contrast to quasi-exact solvability, is used to include the case where the system admits non-normalizable solutions in closed form) [13]. Up to now, three different families of $\mathcal{N}$-fold supersymmetric systems have been found for arbitrary finite integer $\mathcal{N}$, namely, type A [19,20], type B [21], and type C [22], which have correspondence with the classification of second-order linear differential operators preserving a monomial-type vector space [23]. Of the three types, type A and type C under a specific condition have been completely classified [20,22].

While there are still a lot to be done in the mathematical developments of $\mathcal{N}$-fold SUSY, it is also interesting that one looks for physical systems which possess such generalized SUSY. In view of the fact that so far all the $\mathcal{N}$-fold SUSY potentials are only one-dimensional, it is natural that one should look for physical models which are effectively one-dimensional. Experience gained in the work in [6,7] suggests that Pauli and Dirac equations are good candidates to start with. In this respect we note that the authors of [24] found that if the Pauli equation is generalized such that the gyromagnetic ratio $g = 2$ of the electron is changed to some unphysical values $g = 2\mathcal{N}$ ($\mathcal{N} \geqslant 2$), then for certain forms of magnetic fields, the generalized Pauli equation could possess type A $\mathcal{N}$-fold SUSY. The result is interesting in a purely mathematical view point, but unfortunately it would not describe any existing physical systems.

Therefore, we would like to extend the realistic systems in [6,7] to include $\mathcal{N}$-fold SUSY. Since the Pauli and Dirac equations considered there all possess ordinary SUSY, it is therefore natural that we look for an ordinary supersymmetric system which has $\mathcal{N}$-fold supersymmetry as well, as a starting point of the aforementioned purpose. This naturally led us further to consider a more general situation in which a system has simultaneous $\mathcal{N}$-fold supersymmetry with two different values of $\mathcal{N}$. In