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Dynamics of Robertson–Walker spacetimes with diffusion

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ABSTRACT

We study the dynamics of spatially homogeneous and isotropic spacetimes containing a fluid undergoing microscopic velocity diffusion in a cosmological scalar field. After deriving a few exact solutions of the equations, we continue by analyzing the qualitative behavior of general solutions. To this purpose we recast the equations in the form of a two dimensional dynamical system and perform a global analysis of the flow. Among the admissible behaviors, we find solutions that are asymptotically de-Sitter both in the past and future time directions and which undergo accelerated expansion at all times.

1. Introduction

In a recent paper [1] one of us (SC) introduced a new type of fluid matter model in general relativity, in which the fluid particles undergo microscopic velocity diffusion in a cosmological scalar field. The energy–momentum tensor $T^{\mu\nu}$ and the current density J^{μ} of the fluid are given by the usual

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expressions

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu), \qquad J^\mu = n u^\mu, \tag{1}$$

where ρ is the rest-frame energy density, p the pressure, u the four-velocity and n the particle density of the fluid. The diffusion behavior is imposed by postulating the equations

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu, \tag{2a}$$

$$\nabla_\mu (n u^\mu) = 0, \tag{2b}$$

where $\sigma > 0$ is the diffusion constant, which measures the energy gained by the particles per unit of time due to the action of the diffusion forces. Eq. (2b) expresses the usual conservation of the total number of fluid particles, while (2a) is the diffusion equation. In fact it was shown in [2,1] that (2a) is the (formal) macroscopic limit of a Fokker–Planck equation on the kinetic particle density, which is a standard model for diffusion dynamics taking place at the microscopic level on the particles velocity [3].

By projecting (2a) along the direction of u^μ and onto the hypersurface orthogonal to u^μ, we obtain the following equations on the matter fields:

$$\nabla_\mu (\rho u^\mu) + p \nabla_\mu u^\mu = \sigma n, \tag{3a}$$

$$(\rho + p) u^\mu \nabla_\mu u^\nu + u^\nu u^\mu \nabla_\mu p + \nabla^\nu p = 0, \tag{3b}$$

$$\nabla_\mu (n u^\mu) = 0. \tag{3c}$$

The system (3) on the matter variables may be completed by assigning an equation of state between the pressure and the energy density (barotropic fluid). In this paper we assume that

$$p = (\gamma - 1)\rho, \tag{4}$$

for some constant $2/3 < \gamma < 2$, so that in particular the fluid satisfies the strong and dominant energy conditions. The case $\gamma = 1$ gives rise to a dust fluid, while $\gamma = 4/3$ corresponds to a radiation fluid.

Since the energy–momentum is not divergence-free, the coupling of the Einstein equations with the diffusion equation (2a) is incompatible with the (contracted) Bianchi identity

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0. \tag{5}$$

This incompatibility can be resolved by postulating the existence of an additional matter field interacting with the fluid particles and restoring the local conservation of energy. This new matter field plays the role of the background medium in which the particles undergo diffusion. The simplest model for this medium is a vacuum-energy source described by a cosmological scalar field, which leads to the following Einstein equations (in units $8\pi G = c = 1$):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \phi g_{\mu\nu} = T_{\mu\nu}. \tag{6}$$

The evolution equation on the scalar field ϕ determined by (6), the Bianchi identity (5), and the diffusion equation (2a) is

$$\nabla_\nu \phi = \sigma J_\nu. \tag{7}$$

Notice that when $\sigma = 0$, the model under study reduces to the Einstein-perfect fluid system with cosmological constant. The latter has important applications in cosmology, where, under appropriate symmetry assumptions, leads to the ΛCDM model, currently the most popular cosmological model of the universe. The analogous cosmological model with diffusion is the ϕCDM model, in which Λ is replaced by the "variable cosmological constant" ϕ, representing a dark energy field interacting with dark matter by diffusion. This cosmological model has been studied in detail in [4], where in particular an upper bound on the diffusion constant σ has been estimated in order for the model to be compatible with the current cosmological observations. We emphasize that the interaction between the dark