Annals of Physics 354 (2015) 565–569

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

General Navier–Stokes-like momentum and mass-energy equations

Jorge Monreal

Department of Physics, University of South Florida, Tampa, FL, USA

ARTICLE INFO

Article history:
Received 11 July 2014
Accepted 24 January 2015
Available online 2 February 2015

Keywords:
Navier–Stokes
Electromagnetism
Euler equations
Hydrodynamics
Electromagnetic

ABSTRACT

A new system of general Navier–Stokes-like equations is proposed to model electromagnetic flow utilizing analogues of hydrodynamic conservation equations. Such equations are intended to provide a different perspective and, potentially, a better understanding of electromagnetic mass, energy and momentum behaviour. Under such a new framework additional insights into electromagnetism could be gained. To that end, we propose a system of momentum and mass-energy conservation equations coupled through both momentum density and velocity vectors.

1. Introduction

1.1. System of Navier–Stokes equations

Several groups have applied the Navier–Stokes (NS) equations to electromagnetic (EM) fields through analogies of EM field flows to hydrodynamic fluid flow. Most recently, Boriskina and Reinhard made a hydrodynamic analogy utilizing Euler's approximation to the Navier–Stokes equations in order to describe their concept of Vortex Nanogear Transmissions (VNT), which arise from complex electromagnetic interactions in plasmonic nanostructures [1]. In 1998, H. Marmanis published a paper that described hydrodynamic turbulence and made direct analogies between components of the NS equations and Maxwell's equations of electromagnetism[2]. Kambe formulated equations of compressible fluids using analogous Maxwell's relation and the Euler approximation to the NS equations [3]. Lastly, in a recently published paper John B. Pendry, et al. developed a general hydrodynamic model approach to plasmonics [4].

E-mail address: jmonreal@mail.usf.edu.

http://dx.doi.org/10.1016/j.aop.2015.01.026
0003-4916/

In the cases of Kambe and Boriskina, et al., the groups built their models through analogous Euler-like equations along with relevant mass continuity analogues, respectively shown below.

$$\frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} = -\frac{\nabla p}{\rho}, \tag{1}$$

$$\frac{\mathcal{D}\rho}{\mathcal{D}t} + \rho\nabla\cdot\mathbf{v} = 0 \tag{2}$$

where $\mathbf{v}$ is the velocity vector, $\nabla = \frac{\partial}{\partial x_i}\hat{e}_i$ is the del operator, p is pressure, ρ is fluid density, and $\mathcal{D}/\mathcal{D}t = \partial/\partial t + \mathbf{v}\cdot\nabla$ is a material derivative operator. Marmanis and others [5,6] utilized the Navier–Stokes momentum equation (3) to build their EM analogues:

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}. \tag{3}$$

The terms on the left side of the equation represent the fluid's inertia per volume. The $\frac{\partial\mathbf{v}}{\partial t}$ term represent an unsteady state acceleration, while $\mathbf{v}\cdot\nabla\mathbf{v}$ is a non-linear advection term. On the right hand side, the sum of the pressure gradient, ∇p, and the viscosity, $\mu\nabla^2\mathbf{v}$, represent the divergence of a stress tensor. Finally, $\mathbf{f}$ represents the sum of all other body forces acting on the system. Eq. (3) is the momentum equation that describes fluid flow, while Eq. (1) is its approximation under zero body forces and inviscid flow, neglecting heat conduction, also termed the Euler approximation.

As others have done, we, likewise, begin with an analogy of hydrodynamic conservation equations mapped to corresponding electromagnetic conservation equations, assuming non-relativistic flow in an isotropic medium, to finally derive a new system of Navier–Stokes-like equations that model electromagnetic flow. This new set of equations could potentially be useful in gaining a different perspective and better understanding of electromagnetic mass, energy, momentum behaviour.

1.2. General momentum, mass, energy conservation hydrodynamic equations

Eq. (3) is not in its most general form to describe fluid momentum. A more general equation is the Cauchy Momentum equation into which one substitutes in an appropriate stress tensor and constitutive relations relative to the problem at hand. Such substitution then leads to the NS momentum equation. Making use of the material derivative operator, the Cauchy Momentum Equation is:

$$\rho\frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} \tag{4}$$

where $\nabla\cdot\boldsymbol{\sigma}$ is the divergence of a stress tensor, which can be further broken down into the sum of a pressure tensor, $-\nabla p$, and a deviatoric tensor, $\nabla\cdot\boldsymbol{\tau}$. So that, $\nabla\cdot\boldsymbol{\sigma} = -\nabla p + \nabla\cdot\boldsymbol{\tau}$. Here we have opted to represent tensors as boldface lower-case Greek letters.[1]

Given the above, the question then becomes: What is necessary to generally define a hydrodynamic model obeying Navier–Stokes-type equations. The answer comes in the form of conservation of momentum, mass and energy. In terms of the material derivative operator these three are:

$$\text{Momentum:}\quad \rho\frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} - \nabla\cdot\boldsymbol{\sigma} - \mathbf{f} = 0 \tag{5}$$

$$\text{Mass:}\quad \frac{\mathcal{D}\rho}{\mathcal{D}t} + \rho\nabla\cdot\mathbf{v} = 0 \tag{6}$$

$$\text{Energy:}\quad \frac{\mathcal{D}S}{\mathcal{D}t} - \frac{Q}{T} = 0 \tag{7}$$

[1] In component form, the stress tensor can be represented as $\sigma_{ij} = \tau_{ij} + \pi\delta_{ij}$, where τ_{ij} is the stress deviator tensor that distorts a volume component, while $\pi\delta_{ij}$ is the volumetric stress tensor that tends to change the volume of a stressed body due to pressure exertion. Thus, to derive the Navier–Stokes momentum equation from the Cauchy momentum equation a stress tensor of the form: $\sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij}$ is used, with $\mu\epsilon_{ij}$ representing the viscosity component and p the pressure.