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Fractional corresponding operator in quantum mechanics and applications: A uniform fractional Schrödinger equation in form and fractional quantization methods

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ABSTRACT

In this paper we use Dirac function to construct a fractional operator called fractional corresponding operator, which is the general form of momentum corresponding operator. Then we give a judging theorem for this operator and with this judging theorem we prove that R–L, G–L, Caputo, Riesz fractional derivative operator and fractional derivative operator based on generalized functions, which are the most popular ones, coincide with the fractional corresponding operator. As a typical application, we use the fractional corresponding operator to construct a new fractional quantization scheme and then derive a uniform fractional Schrödinger equation in form. Additionally, we find that the five forms of fractional Schrödinger equation belong to the particular cases. As another main result of this paper, we use fractional corresponding operator to generalize fractional quantization scheme by using Lévy path integral and use it to derive the corresponding general form of fractional Schrödinger equation, which consequently proves that these two quantization schemes are equivalent. Meanwhile, relations between the theory in fractional quantum mechanics and that in classic quantum mechanics are also discussed. As a physical example,

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we consider a particle in an infinite potential well. We give its wave functions and energy spectrums in two ways and find that both results are the same.

1. Introduction

Schrödinger equation plays a very important role to describe a system in quantum mechanics. In classic quantum mechanics, many methods can be applied to get Schrödinger equation [1,2]. There are three most important quantization schemes among them: One can derive Schrödinger equation using Hamilton–Jacobi equation with Planck constant as part of coefficient plugged in the equation. The second method is to express quantities as operators and use quantum Poisson brackets to achieve the goal. The third one is to use Dirac Path Integral and Feynman Path Integral by generalizing the trajectory in classical mechanics. It can be proved that the above three methods are equivalent in terms of deriving Schrödinger equation [2].

Nowadays, fractional calculus which is generalized from the classical calculus, becomes a more and more popular tool in various fields of science and engineering. In recent years, fractional calculus has been used in quantum mechanics [3–19]. For example, the fractional Schrödinger equation with fractional derivatives has been put forward by many authors [4,5,7,8,15,16]. We need to notice that the forms of fractional Schrödinger equation are different in fractional quantum mechanics. Laskin derived fractional Schrödinger equation by generalizing Feynman Path Integral to Lévy path integral and introducing Riesz fractional derivative [4,5]. Naber directly replaced the first order time derivative with a Caputo fractional derivative to get the fractional Schrödinger equation [16]. Similarly, Wang and Xu [7], and Dong and Xu [8] get the time–space fractional Schrödinger equation by replacing space derivatives and time derivatives with Riesz fractional derivative and Caputo fractional derivative respectively. Sami I. Muslih, et al. derived another time–space fractional Schrödinger by using fractional Lagrangian formulation [15].

These different fractional Schrödinger equations are endowed with different fractional derivatives in different cases, respectively; so we use Dirac function to define a uniform fractional operator and then compute the expectation of quantities of fractional system with this operator to build the uniform form of fractional Schrödinger equation. In this paper, we would like to study a general fractional system mentioned in [4–6,9–11,17–19]. The momentum formula of the system is P^{α}, and Hamiltonian of the system is $H = D_{\beta}P^{\beta} + V$ (in one dimension), where $\beta = 2\alpha$. When $\alpha = 1$, the Hamiltonian is $H = p^2/2m + V$, which is just the classic case. As an application of this method, we can also derive fractional Schrödinger equation based on other famous fractional derivatives like R–L fractional derivative, G–L fractional derivative, and fractional derivative operator based on generalized functions. Meanwhile we can prove that this method is equivalent to that method by using Lévy path integral. This method to get fractional Schrödinger equation has the benefit that we will no longer need to care about which form of fractional derivative we use. Actually we can choose to deal with the most useful fractional derivative according to different problems under some specific conditions.

This paper is organized as follows: In Section 2, we will recall the definitions of some famous fractional derivatives that we will use in the following parts.

In Section 3, we will define a new fractional operator, named fractional corresponding operator, and give a judging theorem of this operator. And then we can get an interesting result by this judging theorem that all those four fractional derivatives, coincide with the fractional corresponding operator operators.

In Section 4, we construct a new fractional quantization method and get the general form of fractional Schrödinger equation. Then we discuss the relations between the general form and classic Schrödinger equation. As a corollary, we get four new fractional Schrödinger equations.

In Section 5, firstly, we prove that Riesz fractional derivative is the fractional corresponding operator. And then we can derive the fractional Schrödinger equation which is derived with Lévy path