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Novel quasi-exactly solvable models with anharmonic singular potentials

Davids Agboola *, Yao-Zhong Zhang

School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

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ABSTRACT

We present new quasi-exactly solvable models with inverse quartic, sextic, octic and decatic power potentials, respectively. We solve these models exactly by means of the functional Bethe ansatz method. For each case, we give closed-form solutions for the energies and the wave functions as well as analytical expressions for the allowed potential parameters in terms of the roots of a set of algebraic equations.

1. Introduction

Since the early works on singular potentials (see e.g. [1–3]), an extensive literature has been developed on the subject. By singular potentials, we mean those potentials $V(r)$ with property $\lim_{r\to 0} r^2V(r) \to \infty$, although sometimes the inverse-square potential is also regarded as singular. The investigation of singular potentials covers a wide range of physical and mathematical interest. In view of the availability of a comprehensive review article by Frank, Land and Spector [4] on singular potentials and applications, here we will only refer to the main points of the topic.

One of the early works that generated much interest in the study of singular potentials was the one by Predazzi and Regge [5], who argued that real world interactions were likely to be highly singular and thus the study of singular potentials rather than regular potentials might be more relevant physically. This was thereafter followed by the applications of the singular potentials $r^{-n}(n > 2)$ in the study of the (p, p) and (p, π) processes in high energy physics [6,7]. The interactions of nucleons with K-mesons and α–α scattering have been reproduced by repulsive singular potentials [8]. As examples from non-relativistic quantum mechanics, the problem of high-energy scattering by strongly singular potentials was investigated by many authors (see e.g. [9–11]). In molecular physics, inter-atomic or intermolecular forces are mostly represented by singular potentials whose parameters are determined

* Corresponding author.
E-mail addresses: davagboola@gmail.com, d.agboola@maths.uq.edu.au (D. Agboola).

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phenomenologically. Examples are the Mie-type potential [12,13] and the Lennard-Jones potential r^{-12} [14] which describe the interaction between diatomic molecules and two non-polar molecules, respectively.

In field theory, the importance of singular potentials emerged from the efforts to find effective potentials of field-theoretic interactions in the Bethe–Salpeter equations. In [15], Bastai, Bertocchi, Fubini, Furlan and Tonin discovered a remarkable relationship between the renormalizability of field theory and the regularity of the effective potential. That is, the effective potentials for non-renormalizable field theories are singular, whereas superrenormalizable and renormalizable field theories give rive to regular and "transition" potentials, respectively. Thus, any new insight gained in the analysis of singular potentials could lead to a better understanding of quantum field theories which are not perturbatively renormalizable.

In this paper, we consider a class of most frequently discussed singular potentials in nonrelativistic quantum mechanics, that is the spherically symmetric, inverse power potentials of the form [4,16,17],

$$V(r)=\sum_{k=0}^{N}\frac{\lambda_k}{r^{\alpha_k}},$$

where $r\in(0,\infty)$, α_k and λ_k are positive real numbers. The corresponding Schrödinger equation for the radial wave function $\Psi(r)$ is given by

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\omega^2r^2+2V(r)\right]\Psi(r)=2E\Psi(r),\quad \omega\geq 0,$$

where $\ell=-1,0,1,\ldots$, and E is the energy eigenvalue. Much of the investigation of singular potentials is concerned with the solutions of this equation. Several approximation techniques are available in the literature for calculating the eigenvalues and wave functions of some inverse-power potentials (see e.g. [18–23]).

We show that at least four classes of singular potentials in the above family are quasi-exactly solvable, i.e. have polynomial solutions [24–26]. Namely, we present new quasi-exactly solvable models with inverse quartic, sextic, octic and decatic power potentials, respectively. We solve these models exactly by using the functional Bethe ansatz method presented in [27]. For each model, we obtain closed-form solutions for the energies and wave functions as well as analytic expressions for the allowed potential parameters in terms of the roots of a set of algebraic (Bethe ansatz) equations. To our best knowledge, our results on the singular quartic and sextic power potentials are largely new, while our results on the inverse octic and decatic power potentials are completely new.

2. Quasi-exactly solvable inverse quartic power potential

The inverse quartic power potential

$$V(r)=\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\frac{d}{r^4},\quad d>0 \tag{2.1}$$

was investigated by Predazzi and Regge [5] to determine analytic properties of the scattering amplitude in the case of a singular potential. The corresponding radial Schrödinger equation is given by

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\omega^2r^2+\frac{2a}{r}+\frac{2b}{r^2}+\frac{2c}{r^3}+\frac{2d}{r^4}\right]\Psi(r)=2E\Psi(r). \tag{2.2}$$

Phenomenologically, this type of singular potential is a very useful form of anharmonicity in physical applications [20].

Here we are interested in finding exact solutions of the Schrödinger equation. To this end, we first extract the appropriate asymptotic behaviour of the wave function $\Psi(r)$ by making a substitution. After a brief inspection of the differential equation, we arrive at the transformation,

$$\Psi(r)=r^{\gamma}\,\exp\left[Ar^2+Br+\frac{D}{r}\right]x(r), \tag{2.3}$$