Annals of Physics 326 (2011) 2091–2107

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Classical mechanics on noncommutative space with Lie-algebraic structure

Yan-Gang Miao [a,b,c,*], Xu-Dong Wang [a], Shao-Jie Yu [a]

[a] School of Physics, Nankai University, Tianjin 300071, People's Republic of China
[b] Department of Physics, University of Kaiserslautern, P.O. Box 3049, D-67653 Kaiserslautern, Germany
[c] Bethe Center for Theoretical Physics and Institute of Physics, University of Bonn, Nussallee 12, D-53115 Bonn, Germany

ARTICLE INFO

Article history:
Received 21 January 2011
Accepted 14 April 2011
Available online 21 April 2011

Keywords:
Classical mechanics
Noncommutative space

ABSTRACT

We investigate the kinetics of a nonrelativistic particle interacting with a constant external force on a Lie-algebraic noncommutative space. The structure constants of a Lie algebra, also called noncommutative parameters, are constrained in general due to some algebraic properties, such as the antisymmetry and Jacobi identity. Through solving the constraint equations the structure constants satisfy, we obtain two new sorts of algebraic structures, each of which corresponds to one type of noncommutative spaces. Based on such types of noncommutative spaces as the starting point, we analyze the classical motion of the particle interacting with a constant external force by means of the Hamiltonian formalism on a Poisson manifold. Our results *not only* include that of a recent work as our special cases, *but also* provide new trajectories of motion governed mainly by marvelous extra forces. The extra forces with the unimaginable $t\dot{x}-$, $(\dot{x}x)$-, and $(\ddot{x}x)$-dependence besides with the usual t-, x-, and $\dot{x}$-dependence, originating from a variety of noncommutativity between different spatial coordinates and between spatial coordinates and momenta as well, deform greatly the particle's ordinary trajectories we are quite familiar with on the Euclidean (commutative) space.

1. Introduction

Physics founded on noncommutative spacetimes has developed rapidly since the end of last century when the idea of spacetime noncommutativity [1] was revived in the field of string theory [2].

* Corresponding author at: School of Physics, Nankai University, Tianjin 300071, People's Republic of China.
E-mail addresses: miaoyg@nankai.edu.cn (Y.-G. Miao), c_d_wang@mail.nankai.edu.cn (X.-D. Wang), leptonyu@gmail.com (S.-J. Yu).

0003-4916/$ - see front matter
doi:10.1016/j.aop.2011.04.009

There have been a large amount of literature on noncommutative quantum mechanics (NCQM) [3] and noncommutative field theory (NCFT) [4] as well. However, we notice that the research on noncommutative classical mechanics[1] (NCCM) [5–7], if comparing with that of NCQM and NCFT, is so little. Probably the NCCM is not so attractive as the NCQM and NCFT; nevertheless, the Doubly Special Relativity [7] has been intriguing. Moreover, the present paper tries to give from the nonrelativistic aspect a glance at a variety of interesting properties that the NCCM possesses.

The mathematical background for the physics on noncommutative spacetimes is the noncommutative geometry [8]. As was demonstrated, *e.g.* in Ref. [5], the spacetime noncommutativity can be distinguished into three kinds in accordance with the Hopf-algebraic classification [9], that is, there exist the canonical, Lie-algebraic and quadratic noncommutativity, respectively. In addition, the three types of noncommutative spacetimes have been studied in the framework of quantum groups at both the nonrelativistic and relativistic levels, and the relative Hopf algebras for some specific noncommutative spacetimes[2] have been given [10–15]. In brief, at the former level the Galilei Hopf algebras have been provided for the canonical [10,12], Lie-algebraic [10–12] and quadratic [12] noncommutativity, respectively; and at the latter level the Poincaré Hopf algebras have been proposed for the canonical [13,10], Lie-algebraic [14,10,11] and quadratic [15] noncommutativity, respectively. Incidentally, the Hopf algebras for general noncommutative spacetimes still remain unknown.[3]

In a recent work [5] a nonrelativistic particle interacting with a constant external force on the (Lie-algebraic and quadratic) noncommutative phase spaces with commutative momenta was dealt with in detail and some intriguing trajectories were then revealed. This classical system is independent of any star-products and can thus be analyzed by means of the Hamiltonian formalism on a Poisson manifold. The reason lies both on the constant external force and on the commutative momenta (see the discussions in the next section for the details). If the external force were not constant and the momenta were not commutative, one would have to envisage the corresponding star-product rules which are complicated and unknown at present. Therefore, Ref. [5] provides an alternative way to study a simplified classical system on a kind of specific (Lie-algebraic and quadratic) noncommutative spaces of which the rules of star-products are unknown. In other words, one is able to analyze some simple classical motion, such as that associated with a constant external force, on so complicated noncommutative spaces that their corresponding star-product rules are even unknown. However, we would like to point out that the (Lie-algebraic) noncommutative spaces considered in Ref. [5] as the starting point for investigating the particle's noncommutative kinetics are special cases of our findings.

In this paper we generalize Ref. [5] to more complicated Lie-algebraic noncommutative spaces. Our generalization focuses only on the Poisson brackets between different spatial coordinates and between spatial coordinates and momenta, but still keeps the Poisson brackets between different momenta vanishing. See Eqs. (8) and (9). That is, we consider our generalization within the limitation that no star-products are needed, which can be seen clearly in the Newton equations, Eqs. (23)–(25) and (38)–(40), where no products of spatial coordinates associated with non-vanishing Poisson brackets exist. To this end, we propose a practical and effective way to look for more complicated Lie-algebraic noncommutative spaces than that appeared in Ref. [5], which is in fact based on Lie-algebraic properties, such as the antisymmetry and Jacobi identity. In other words, we solve the constraint equations that the structure constants of a Lie algebra, here known as noncommutative parameters, hold, and obtain two new types of noncommutative spaces with the Lie-algebraic structure which cover that of Ref. [5] as subclasses. In terms of the Hamiltonian analysis on a Poisson manifold [16], we derive the Hamilton equation and Newton equation, and observe that the noncommutativity between different spatial coordinates and between spatial coordinates and momenta provides various extra forces that make the particle's ordinary trajectories on the Euclidean (commutative) space have a quite large deformation. We point out, as a byproduct, that the Lie-algebraic noncommutative spaces are anisotropic because of the direction-dependent deformation.

[1] Here the classical mechanics includes both the nonrelativistic and relativistic mechanics.

[2] For example, the κ-deformed Minkowski spacetime [14] as a specific case of the Lie-algebraic noncommutativity has recently been paid much attention because it is a natural candidate for the spacetime based on which the Doubly Special Relativity [7] has been established.

[3] For instance, it may be a possible progress to construct the Hopf algebras for Type I and Type II spaces (Eqs. (8) and (9)).