Annals of Physics 326 (2011) 2174–2185

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Algebraic structure of general electromagnetic fields and energy flow

Shahen Hacyan *

Instituto de Física, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, México D.F. 01000, Mexico

ARTICLE INFO

Article history:
Received 10 March 2011
Accepted 5 April 2011
Available online 8 April 2011

Keywords:
Electromagnetic field
Algebraic structure
Energy–momentum
Bessel beams

ABSTRACT

The algebraic structures of a general electromagnetic field and its energy–momentum tensor in a stationary space–time are analyzed. The explicit form of the reference frame in which the energy of the field appears at rest is obtained in terms of the eigenvectors of the electromagnetic tensor and the existing Killing vector. The case of a stationary electromagnetic field is also studied and a comparison is made with the standard short-wave approximation. The results can be applied to the general case of a structured light beams, in flat or curved spaces. Bessel beams are worked out as example.

1. Introduction

Structured electromagnetic waves, such as Gauss, Laguerre, Airy, or Bessel beams, are naturally used in all optics experiments (see e.g., [1–6]). From a theoretical point of view, a structured field can be analyzed decomposing it into plane-wave modes. However, this well established method has certain limitations because the algebraic structure of plane-waves is particularly simple. Thus, for instance, the quantization of a structured electromagnetic field does not yield all the quantum operators with the usual set of commutation relations, as is the case of Bessel beams [7].

As for the conservations laws, the background space–time must have some symmetry properties in order to define conserved quantities. In Minkowski space, the Poincaré group provides such symmetry. In a more general curved space–time, the existence of some Killing vectors is necessary; in particular, a time-like Killing vector is required for an unambiguous definition of energy and its conservation.

The electromagnetic field tensor admits two scalar quantities that permit its classification into either null or general fields (see e.g., [8,9]). These two scalars vanishes for plane-waves, but not for

* Tel.: +52 5556225137; fax: +52 5556225015.
E-mail address: hacyan@fisica.unam.mx

0003-4916/$ - see front matter
doi:10.1016/j.aop.2011.04.006

structured fields such as, for instance, Bessel beams [6]. The aim of the present paper is to analyze the algebraic structure of a general field and its associated energy–momentum tensor, assuming that the background space–time admits a time-like Killing vector. A covariant form of the energy–momentum four-vector is obtained in terms of the eigenvectors of the electromagnetic tensor and the Killing vector. On the basis of this analysis, it is pointed out, that except for null electromagnetic fields, the energy of the field does not flow along null world-line; thus the Poynting vector cannot be identified with the momentum of a massless particle. This may be at the root of the difficulties in quantizing an electromagnetic field beyond the plane-wave approximation.

The plan of this article is as follows. In Section 2, the natural tetrad associated to the electromagnetic field is obtained in explicit form. Section 3 is devoted to the definition of energy and momentum in a space–time admitting a time-like Killing vector. In Section 4, a stationary electromagnetic field (equivalently a monochromatic wave) is studied and the connection with the usual treatment is made through the short-wave approximation. The results are obtained in a fully covariant form and can be applied to a general relativistic study. The particular case of a Minkowski space is presented in more details in Section 5, together with an application to Bessel beams as an illustrative example. The concept of the photon four-momentum is briefly discussed in the last Section.

2. General formulation

The electromagnetic field is defined by an antisymmetric tensor $F_{\alpha\beta}$. Two scalars of the field can be conveniently defined as

$$F^{\alpha\beta}F_{\alpha\beta} \equiv 2(\mathcal{B}^2 - \mathcal{E}^2), \tag{2.1}$$

$$F^{\alpha\beta}F^*_{\alpha\beta} \equiv 4\mathcal{E}\mathcal{B}, \tag{2.2}$$

where the dual tensor of the field is

$$F^*_{\alpha\beta} \equiv \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$$

and $\mathcal{E}$ and $\mathcal{B}$ are scalar functions. Here and in the following, the line element of the background space–time is taken as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the metric tensor and g is its determinant. Throughout this paper, we use the signature $(-+++)$ and set $c = 1$.

The energy–momentum tensor of the field is

$$T^{\alpha\beta} = -F^{\alpha\cdot}_{\mu}F^{\mu\beta} - \frac{1}{4}g^{\alpha\beta}(F^{\lambda\mu}F_{\lambda\mu}), \tag{2.3}$$

in Heaviside–Lorentz units. The conservation of energy and momentum, in the absence of charges and currents, is given by the condition

$$T^{\alpha\beta}_{;\beta} = 0, \tag{2.4}$$

which follows from the Maxwell equations.

An elegant algebraic description of the electromagnetic field tensor was elaborated by Plebanski [9]. Since the original reference is not easily accessible, we repeat the basic results in this section for the sake of completeness. The starting point is the definition of the matrix

$$\mathbb{F} \equiv \{F^\alpha_\beta\},$$

with eigenvectors $k \equiv \{k^\alpha\}$ given by

$$\mathbb{F}k = \lambda k \tag{2.5}$$

and where the eigenvalue λ satisfies the equation

$$\lambda^4 + (\mathcal{B}^2 - \mathcal{E}^2)\lambda^2 - \mathcal{E}^2\mathcal{B}^2 = 0. \tag{2.6}$$

Due to the antisymmetry of $F^{\alpha\beta}$, the vector k^α is a null-vector: