Available online at www.sciencedirect.com
ScienceDirect
Annals of Physics 323 (2008) 893–906
ANNALS of PHYSICS
www.elsevier.com/locate/aop

Linear transformations of quantum states

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Received 18 May 2007; accepted 12 June 2007
Available online 16 June 2007

Abstract

This paper considers the most general linear transformation of a quantum state. We enumerate the conditions necessary to retain a physical interpretation of the transformed state: hermiticity, normalization and complete positivity. We show that these can be formulated in terms of an associated transformation introduced by Choi in 1975. We extend his treatment and display the mathematical argumentation in a manner closer to that used in traditional quantum physics. We contend that our approach displays the implications of the physical requirements in a simple and intuitive way. In addition, defining an arbitrary vector, we may derive a probability distribution over the spectrum of the associated transformation. This fixes the average of the eigenvalue independently of the vector chosen. The formal results are illustrated by a couple of examples.

PACS: 03.65.−w; 03.65.Ta

Keywords: Linear transformations; Complete positivity; Kraus sum

1. Introduction

The most general representation of the state of a quantum system is the density matrix ρ_{kl}. Physical modifications of this take the form of linear transformations; nonlinear

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0003-4916/$ - see front matter
doi:10.1016/j.aop.2007.06.001

transformations make it hard to retain the interpretation of the density matrix as a probability distribution on state vectors. Both time evolution and observations lead to linear transformations, which thus are ubiquitous in physics. Even more general quantum systems can be defined by their transformation of input states into outputs. It is consequently of interest to enquire what form transformations may take and which are the criteria for accepting them.

Such considerations have a long history. Both physicists and mathematicians have investigated the applications of linear transformations. Their mathematical properties are presented in [1]. Such transformations, on the other hand, have acquired intense interest recently in the context of dealing with quantum information [2].

The probability interpretation of the density matrix requires the transformations to be positive even when entangled with another system; this is complete positivity (CP). The solution to this problem has been known for a long time [1]; a so-called Kraus form is necessary and sufficient. It is thus of interest to test when a linear transformation is CP. A convenient form of the necessary and sufficient conditions was put forward in [3,4]. The latter publication has become the standard reference to this treatment. It is essentially repeated in many later works such as [5,6], see also the references in these papers.

Another approach to the question of complete positivity takes various time evolution equations and investigates if and when they obey the positivity requirement. Such papers include [7–9]; further examples can be found in the references to these works.

In [4], Choi finds a sufficient and necessary condition that a given linear transformation is CP in terms of the linear transformation operating on the basis states of the density matrix. A slightly different argument is provided by Caves in [6], who utilizes the possibility to purify the density matrix by an extension of the state space. The method is applied to a problem in quantum dynamics by Andersson et al. in [10].

The criterion formulated by Choi is widely applicable, because it requires only considerations carried out in the framework where the transformation is operating. It utilizes no considerations of environment degrees of freedom, whether physical or formally introduced. Thus, it is the most expedient way to test and classify linear transformations. Unfortunately its introduction is purely formal and does not illuminate the structure underlying its validity. Ref. [11] claims to make the Choi result more acceptable to physicists, but it seems to us that its presentation still emerges as a very formal procedure perhaps lacking intuitive appeal.

Quantum mappings have recently become significant in information transmission and processing. They both propose and restrict the technical solutions to be applied in physical systems. Unfortunately the theoretical treatment in the field has acquired mathematical parlance and rigour of methods beyond that usually employed in quantum mechanics. The compactness of the notation and the precision of the argumentation make it arduous to follow for physicists. In particular, it becomes hard to grasp intuitively the necessity of the various aspects of the results. In this paper, we try to reformulate the basic conclusions concerning linear mappings of states within a framework more familiar to quantum physicists. In many aspects, we reproduce the results of the more mathematical treatments, but our methods differ considerably from those in the literature. We do find some new results however, which follow from our approach. We hope that the methods and results presented may serve to elucidate the profound properties and central significance of allowed linear mappings of quantum states.