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How to introduce time operator

Zhi-Yong Wang *, Cai-Dong Xiong

School of Physical Electronics, University of Electronic Science and Technology of China, Chengdu 610054, China

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Abstract

Time operator can be introduced by three different approaches: by pertaining it to dynamical variables; by quantizing the classical expression of time; and taken as the restriction of energy shift generator to the Hilbert space of a physical system.

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1. Introduction

Traditionally, time enters quantum mechanics as a parameter rather than a dynamical operator. As a consequence, the investigations on tunneling time, arrival time and traversal time, etc., still remain controversial today [1–19]. On the one hand, one imposes self-adjointness as a requirement for any observable; on the other hand, according to Pauli's argument [20–23], there is no self-adjoint time operator canonically conjugating to a Hamiltonian if the Hamiltonian spectrum is bounded from below. A way out of this dilemma set by Pauli's objection is based on the use of positive operator valued measures (POVMs) [19,22–26]: quantum observables are generally positive operator valued measures, e.g., quantum observables are extended to maximally symmetric but not

* Corresponding author.
E-mail address: zywang@uestc.edu.cn (Z.-Y. Wang).

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necessarily self-adjoint operators, in such a way one preserves the requirement that time operator be conjugate to the Hamiltonian but abandons the self-adjointness of time operator.

In this paper, general time operators are constructed by three different approaches. In the following, the natural units of measurement ($\hbar = c = 1$) is applied.

2. Mandelstam–Tamm version of non-self-adjoint time operator

The time–energy uncertainty relation has been a controversial issue and has many versions. However, the Mandelstam–Tamm version [27] of the time–energy uncertainty is the most widely accepted nowadays, where the time deviation ΔT is given by a characteristic time associated with some dynamical variables. Similarly, we will show that time operator can be introduced by pertaining it to dynamical variables. Let $\hat{F}$ be a non-stationary observable (but does not depend explicitly on time t: $\partial\hat{F}/\partial t = 0$), in Heisenberg picture it satisfies the Heisenberg equation of motion for $\hat{F}$

$$\mathrm{i}\mathrm{d}\hat{F}/\mathrm{d}t = [\hat{F}, \hat{H}]. \tag{1}$$

Assume that whenever $\mathrm{d}\hat{F}/\mathrm{d}t$ is nonzero such that it has the inverse

$$(\mathrm{d}\hat{F}/\mathrm{d}t)^{-1} = \mathrm{i}([\hat{F}, \hat{H}])^{-1}. \tag{2}$$

If $[(\mathrm{d}\hat{F}/\mathrm{d}t)^{-1}, \hat{H}] = 0$, i.e., $[([\hat{F}, \hat{H}])^{-1}, \hat{H}] = 0$, one can introduce a time operator $\hat{T}_1$ as follows:

$$\begin{aligned}\hat{T}_1 &= -[\hat{F}(\mathrm{d}\hat{F}/\mathrm{d}t)^{-1} + (\mathrm{d}\hat{F}/\mathrm{d}t)^{-1}\hat{F}]/2\\ &= -\mathrm{i}[\hat{F}([\hat{F}, \hat{H}])^{-1} + ([\hat{F}, \hat{H}])^{-1}\hat{F}]/2\end{aligned} \tag{3}$$

Using Eqs. (1)–(3) and $[([\hat{F}, \hat{H}])^{-1}, \hat{H}]$=0, one has

$$[\hat{H}, \hat{T}_1] = \mathrm{i}, \tag{4}$$

i.e., $\hat{T}_1$ and $\hat{H}$ form a canonically conjugate pair. Owing to the fact that the Hamiltonian spectrum is bounded from below, the time operator $\hat{T}_1$ satisfying Eq. (4) is not self-adjoint. However, according to the formalism of POVMs [21–26], it can represent an observable.

As an example, assume that in (1+1) dimensional space-time (t, x), a freely moving particle (with mass m) has position x and momentum $\hat{p} = -\mathrm{i}\partial/\partial x$, and its the Hamiltonian is denoted as $\hat{H} = \hat{p}^2/2m$. Let $\hat{F} = x$, applying Eq. (3) and $[x, \hat{p}] = \mathrm{i}$, one has

$$\hat{T}_1 = -m(\hat{p}^{-1}x + x\hat{p}^{-1})/2, \tag{5}$$

or in the momentum representation,

$$\hat{T}_1 = -\frac{\mathrm{i}m}{2}\left(\frac{1}{p}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\frac{1}{p}\right), \tag{6}$$

it is the free-motion time-of-arrival operator that has been studied in many literatures (see for example, Refs.[11,19,21–26]). For the time being, the time operator is defined on a dense domain of the Hilbert space of square integrable functions on the half real-line [11,19,21–26], denoted as $\mathcal{D}(\hat{T}) = \mathcal{H}_0 = L^2(\mathcal{R}^+, \mathrm{d}E)$, where E corresponds to the eigenvalue of $\hat{H} = \hat{p}^2/2m$ and $\mathcal{R}^+ = \{E \mid 0 \leqslant E < +\infty\}$.