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An analytic function approach to weak mutually unbiased bases

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ABSTRACT

Quantum systems with variables in $\mathbb{Z}(d)$ are considered, and three different structures are studied. The first is weak mutually unbiased bases, for which the absolute value of the overlap of any two vectors in two different bases is $1/\sqrt{k}$ (where $k|d$) or 0. The second is maximal lines through the origin in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space. The third is an analytic representation in the complex plane based on Theta functions, and their zeros. It is shown that there is a correspondence (triality) that links strongly these three apparently different structures. For simplicity, the case where $d = p_1 \times p_2$, where p_1, p_2 are odd prime numbers different from each other, is considered.

1. Introduction

After the pioneering work by Schwinger [1], there has been a lot of work on various aspects of a quantum system $\Sigma(d)$ with variables in $\mathbb{Z}(d)$ (the ring of integers modulo d), described with a d-dimensional Hilbert space $H(d)$. The work combines Quantum Physics with Discrete Mathematics and has applications to areas like quantum information, quantum cryptography, quantum coding, etc. (for reviews see [2–8]).

A deep problem in this area is mutually unbiased bases [9–18]. It is a set of bases, for which the absolute value of the overlap of any two vectors in two different bases is $1/\sqrt{d}$. It is known that the number $\mathfrak{M}$ of mutually unbiased bases satisfies the inequality $\mathfrak{M} \leq d + 1$, and that when d is a prime number $\mathfrak{M} = d + 1$. What makes the case of prime d special, is that $\mathbb{Z}(d)$ becomes a field, which is

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a stronger mathematical structure than a ring. For the same reason, if we consider quantum systems with variables in the Galois field $GF(p^e)$ (where p is a prime number), the number of mutually unbiased bases is $\mathfrak{M} = p^e + 1$. The study of mutually unbiased bases for non-prime d, in which case $\mathbb{Z}(d)$ is a ring (but not a field), is a very difficult problem. It is also related to the subjects of t-designs [19,20] and latin squares [21].

Recent work [22,23] introduced a weaker concept called weak mutually unbiased bases (WMUB). It is a set of bases, for which the absolute value of the overlap of any two vectors in two different bases is $1/\sqrt{k}$, where $k|d$ (k is a divisor of d), or zero. It has been shown that there are $\psi(d)$ (the Dedekind ψ-function) WMUBs. This work has also studied the phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$ as a finite geometry $\mathcal{G}(d)$.

There exists much literature on finite geometries. They consist of a finite number of points and lines which obey certain axioms (e.g., [24–26] in a mathematics context, and [27–30] in a physics context). Most of this work is on near-linear geometries, where two lines have at most one point in common. The $\mathbb{Z}(d) \times \mathbb{Z}(d)$ geometry is based on rings and it does not obey this axiom. Two lines have in common a 'subline' which consists of k points, where $k|d$. Refs. [22,23] have shown that there is a duality between WMUBs in $H(d)$ and lines in $\mathcal{G}(d)$. This shows a deep connection between finite quantum systems and the geometries of their phase spaces.

A very different problem is the use of analytic functions in the context of physical systems. After the pioneering work by Bargmann [31,32] for the harmonic oscillator, analytic representations have been used with various quantum systems (e.g., [33–42]). In particular the zeros of the analytic functions have been used for the derivation of physical results. For example, there are links between the growth of analytic functions at infinity, and the density of their zeros [43–45], which lead to criteria for the overcompleteness or undercompleteness of a von Neumann lattice of coherent states.

Refs. [46,47] have studied analytic representations for quantum systems with variables in $\mathbb{Z}(d)$, using Theta functions [48] (see also Ref. [49]). Quantum states are represented with analytic functions in the cell $\mathfrak{S} = [0, d) \times [0, d)$ in the complex plane (i.e., in a torus). These analytic functions have exactly d zeros in the cell $\mathfrak{S}$, which determine uniquely the state of the system.

In this paper we use this language of analytic functions for the study of WMUBs. We show that:

- Each of the d vectors in a WMUB has d zeros on a straight line.
- In a given WMUB, the various vectors have zeros on parallel lines. In different WMUBs, the slope of the lines of zeros, is different.
- The d^2 zeros in each WMUB, form a regular lattice in the cell $\mathfrak{S}$, which is the same for all WMUBs.

Based on these results we show that there is a triality between

- WMUBs.
- Lines through the origin in the finite geometry $\mathcal{G}(d)$ of the phase space.
- Sets of parallel lines of zeros of the vectors in WMUBs in the cell $\mathfrak{S}$.

These three mathematical objects, which are very different from each other, have the same mathematical structure. The work links the theory of analytic functions and their zeros, to finite quantum systems, finite geometries and more generally to Discrete Mathematics.

In order to avoid a complicated notation, in all sections except Section 2, we consider the case that $d = p_1 \times p_2$, where p_1, p_2 are odd prime numbers, different from each other (in Section 2 we state in each subsection, what values d takes). All results are generalizable to the case $d = p_1 \times \cdots \times p_N$, where d is an odd integer (see discussion). In the case of even dimension d (e.g., [50]), some aspects of the formalism of finite quantum systems require special consideration, and further work is needed in order to extend the ideas of the present paper, to this case. Also when d contains powers of prime numbers, further work is needed (based on labelling with elements of Galois fields).

In Section 2 we introduce very briefly finite quantum systems, their analytic representation, and mutually unbiased bases, in order to define the notation. In Section 3 we review briefly the formalism of weak mutually unbiased bases. An important ingredient is the factorization of $\Sigma(d)$ in terms of smaller systems $\Sigma(p_1)$ and $\Sigma(p_2)$, which is based on the Chinese remainder theorem, and its use by Good [51] in the context of finite Fourier transforms. In Section 4, we use the analytic representation to study WMUBs, and prove the results that we mentioned above. We conclude in Section 5, with a discussion of our results.