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From liquid crystal models to the guiding-center theory of magnetized plasmas

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ABSTRACT

Upon combining Northrop's picture of charged particle motion with modern liquid crystal theories, this paper provides a new description of guiding center dynamics (to lowest order). This new perspective is based on a rotation gauge field (*gyrogauge*) that encodes rotations around the magnetic field. In liquid crystal theory, an analogue rotation field is used to encode the rotational state of rod-like molecules. Instead of resorting to sophisticated tools (e.g. Hamiltonian perturbation theory and Lie series expansions) that still remain essential in higher-order gyrokinetics, the present approach combines the WKB method with a simple kinematical ansatz, which is then replaced into the charged particle Lagrangian. The latter is eventually averaged over the gyrophase to produce the guiding-center equations. A crucial role is played by the vector potential for the gyrogauge field. A similar vector potential is related to liquid crystal defects and is known as *wryness tensor* in Eringen's micropolar theory.

1. Introduction

This paper aims to establish some analogies between liquid crystal models and the theory of guiding center motion in plasma physics. Evidently, these two theories arose in very different fields of physics and their relationships are far from obvious. However, this paper shows that the gyroradius of Northrop's guiding center picture [1,2] and the director field of uniaxial liquid crystal molecules (in both the Ericksen–Leslie and Eringen's formulations) possess several similarities that can be exploited

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to shed new light on Littlejohn's geometric use of gyro-gauge potentials [3]. Indeed, the concept of rotational state of a rod-like nematic molecule can be transferred to define a rotational state of the gyroradius. The essential difference between the two concepts is that the gyroradius can only rotate around the magnetic field vector and this is encoded by using the axis–angle parametrization of three-dimensional rotations (Rodrigues' formula). Despite this difference, the two concepts are related by common geometric features, which can be exploited by the use of rotation matrix fields. In turn, this leads in both cases to the emergence of a certain gauge potential that occurs in Eringen's micropolar theory [4] and is a key geometric object in Littlejohn's theory of guiding center motion [3,5–7].

These analogies can be pushed even further by considering biaxial nematic molecules (as described by a pair of orthogonal directors) and the pair of unit vectors associated to the magnetic field and the gyroradius. This analogy is particularly transparent for a time-dependent magnetic field, although in this case guiding center theory requires splitting rotations around the magnetic field from rotations of the magnetic field itself. This splitting introduces a composition of two- and three-dimensional rotations that leads to an interesting form of the gauge potential.

In order to provide a systematic exposition, we shall first review the essential concepts associated to the rotational state of liquid crystal molecules. Then, this introduction will proceed to illustrate the main ideas underlying Northrop's guiding center picture. The main analogies will also be discussed.

1.1. Uniaxial nematic phases and the director field

Liquid crystals are well known examples of fluids with internal (micro-)structure [8,9]. More particularly, liquid crystals are typically modeled as fluids carrying particles that are endowed with an orientational state. This orientational state is incorporated in an additional microscopic variable, that is known in condensed matter theory as *order parameter*. This order parameter emerges when the full rotational symmetry is broken by the particular shape of the particle (or molecule). Several types of liquid crystal phases are available and each of them is modeled by a different type of order parameter in some coset space [10,11]. In the simplest case of nematic liquid crystals, rod-like molecules carry a preferred direction in space. Then, the order parameter is identified with an unsigned unit vector $\mathbf{n}$, called the *director*. In the more complicated case of biaxial nematic molecules, the order parameter is identified with a pair of orthogonal directors.

When one aims to consider the entire liquid crystal flow, one builds a continuum theory by replacing the order parameter of the single molecule by an order parameter field, which is evaluated at the Lagrangian particle position. The most celebrated theory of uniaxial nematic flows was formulated by Ericksen and Leslie [12], who coupled Euler's fluid equation to an Euler–Lagrange equation for the director field $\mathbf{n}(\mathbf{x}, t)$. The same theory can be expressed in terms of the director field $\mathbf{n}(\mathbf{x}, t)$ and its angular velocity field $\boldsymbol{\omega}$ such that $\partial_t \mathbf{n} = \boldsymbol{\omega} \times \mathbf{n}$. One way the angular velocity is naturally incorporated in the treatment is by writing the director dynamics in terms of the rotational state $\mathcal{R}$ as follows:

$$\mathbf{n}(\mathbf{x}, t) = \mathcal{R}(\mathbf{x}, t)\mathbf{n}_0(\mathbf{x}), \tag{1}$$

where $\mathcal{R}$ is a time-dependent rotation gauge field, so that $\mathcal{R}(\mathbf{x}, t)$ is a rotation matrix at all times and at all points in space (such that $\mathcal{R}(\mathbf{x}, 0) = \mathbf{1}$). Here, we have not considered the fluid motion as this is not essential for our purposes. This approach was followed by Dzyaloshinskii and Volovick [13] to formulate a Poisson-bracket structure to uniaxial nematic flows. The same idea was later implemented in the Lagrangian variational framework [14,15]. In the latter case, the angular velocity appears under its skew-symmetric matrix correspondent $\widehat{\omega}$ as follows:

$$\partial_t \mathbf{n} = (\partial_t \mathcal{R})\mathcal{R}^{-1}\mathbf{n} = \widehat{\omega}\mathbf{n}, \tag{2}$$

so that $\widehat{\omega} = (\partial_t \mathcal{R})\mathcal{R}^{-1}$ and the angular velocity vector is given in components by $\omega_a = \varepsilon_{abc}\widehat{\omega}_{cb}/2$.

Then, the rotation field becomes a local gauge field and the whole liquid crystal theory can be formulated as a gauge theory. For example, a gauge vector potential emerges upon writing

$$\nabla_i \mathbf{n} = (\nabla_i \mathcal{R})\mathcal{R}^{-1}\mathbf{n} - \mathcal{R}\widehat{\gamma}_{0\,i}\mathcal{R}^{-1}\mathbf{n},$$