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Quantum spin chains with fractional revival

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ABSTRACT

A systematic study of fractional revival at two sites in *XX* quantum spin chains is presented. Analytic models with this phenomenon are obtained by combining two basic ways of realizing fractional revival in a spin chain. The first proceeds through isospectral deformations of spin chains with perfect state transfer. The second makes use of couplings provided by the recurrence coefficients of polynomials with a bi-lattice orthogonality grid. The latter method leads to analytic models previously identified that can exhibit perfect state transfer in addition to fractional revival.

1. Introduction

Spin chains with engineered couplings have proved attractive for the purpose of designing devices to achieve quantum information tasks such as quantum state transfer (over relatively short distances) or entanglement generation [1–3]. One reason for the interest is that the internal dynamics of the chain takes care of the processes with a minimum of external intervention required. In this perspective, a desired feature of such chains is that they exhibit quantum revival. For perfect state transfer (PST), one wishes to have, for example, a one-excitation state localized at the beginning of the chain evolve with unit probability, after some time T, into the state with the excitation localized at the end of the chain. Such a relocalization of the wave packet is what is referred to as a revival [4,5]. Fractional revival (FR) occurs when a number of smaller packets seen as clones of the original one form at certain

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sites and show local periodicities [4,6]. Its realization in a spin chain would also allow to transport information with high efficiency via one of the clones. Moreover, in a balanced case where there is equal probability of finding clones at the beginning and at the end of a chain, fractional revival would provide a mechanism to generate entangled states. It is hence of relevance to determine if FR is feasible in spin chains, and if so, in which models. Some studies have established the fact that this effect can indeed be observed in spin chains [2,7–9]. The present paper offers a systematic analysis of the circumstances under which fractional revival at two sites in quantum spin chains of the *XX* type will be possible. The main question is to determine the Hamiltonians H that will have the fractional revival property. Like for PST, the conditions for FR are expressed through requirements on the one-excitation spectrum of H. One then deals with an inverse spectral problem that can be solved with the assistance of orthogonal polynomial theory. In the PST case for which this analysis has been carried out in detail (e.g. [10]), one sees that a necessary condition is that the couplings and Zeeman terms must form a three-diagonal matrix that is mirror-symmetric. Furthermore, models based on particular orthogonal polynomials (OPs) have been found in which PST can be exhibited in an exact fashion. These analytic models are quite useful; in fact, the simplest one [11] was employed to perform an experimental quantum simulation with Nuclear Magnetic Resonance techniques of mirror inversion in a spin chain [12]. We shall here also provide analytic models with fractional revival.

The outline of the paper is as follows. In Section 2, we present the Hamiltonians H for the class of *XX* spin chains that will be considered. The reader is reminded that their one-excitation restrictions J are given by Jacobi matrices and the orthogonal polynomials associated to the diagonalization of those J's are described. In Section 3, we review the elements of the characterization of *XX* spin chains with PST that will be essential in our fractional revival study. In particular, the necessary and sufficient conditions for PST in terms of the spectrum of J, the mirror symmetry and the properties of the associated orthogonal polynomials will be recalled.

In Section 4, we undertake a systematic analysis of the conditions under which fractional revival can occur at two sites. Up to a global phase, the revived states will be parametrized in terms of two amplitudes $\sin 2\theta$ and $e^{i\psi}\cos 2\theta$; the case $\theta = 0$ will correspond to the PST situation. It will be shown that in general, for FR to occur, the one-excitation spectrum of the Hamiltonian must take the form of a bi-lattice, i.e. the spectral points need to be the union of two uniform lattices translated one with respect to the other by a parameter δ depending on θ and ψ. Two special cases, namely $\psi = 0$ and $\psi = \pi/2$, will be the object of the subsequent two sections. In the first case the spectrum condition is the PST one and in the second case, it is instead the mirror symmetry that is preserved. These two cases will provide the ingredients of a two-step process for obtaining the most general *XX* chains with FR at two sites.

In Section 5, it will be shown how spin chains with FR can be obtained from isospectral deformations of spin chains with PST (relative phase $\psi = 0$). Analytic models with FR will thus be obtained from analytic models with PST. It will be observed that only the central couplings of the chain will need to be modified so as to make FR happen. Mirror-symmetry will be seen to be replaced by a more complicated inversion condition. The orthogonal polynomials corresponding to the deformed Jacobi matrix will be shown to have a simple expression in terms of the unperturbed orthogonal polynomials associated to the parent PST Hamiltonian. Their knowledge will be relevant to the construction, at the end of Section 6, of the FR Hamiltonian with arbitrary parameters θ and ψ.

Section 6 will deal with the special case $\psi = \pi/2$, that is when one amplitude is real and the other purely imaginary. It will be seen that J is mirror symmetric in this situation with a bi-lattice spectrum. As shall be explained, an exact solution to the corresponding inverse spectral problem turns out to be already available and is provided by the recurrence coefficients of the para-Krawtchouk polynomials introduced in [13]. It is remarkable that these somewhat exotic functions naturally arise in the FR analysis. We shall demonstrate that for certain values of the parameters, the models thus engineered possess both FR and PST. We shall finally come to the determination of the generic Hamiltonian with FR and show that it is obtained from the recurrence coefficients of the para-Krawtchouk polynomials perturbed in the way described in Section 5; in other words it is constructed by performing an isospectral deformation of the Jacobi matrix of the para-Krawtchouk polynomials. The two arbitrary parameters are related to the bi-lattice and to the deformation parameter. We shall summarize the outcome of the analysis and offer final remarks to conclude.