Annals of Physics 367 (2016) 288–296

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Generalized microcanonical and Gibbs ensembles in classical and quantum integrable dynamics

Emil A. Yuzbashyan

Center for Materials Theory, Rutgers University, Piscataway, NJ 08854, USA

ARTICLE INFO

Article history:
Received 16 December 2015
Accepted 1 February 2016
Available online 8 February 2016

Keywords:
Coherent many-body dynamics
Quantum quenches
Statistical ensembles
Integrable systems
Generalized Gibbs Ensemble
Prethermalization

ABSTRACT

We prove two statements about the long time dynamics of integrable Hamiltonian systems. In classical mechanics, we prove the microcanonical version of the Generalized Gibbs Ensemble (GGE) by mapping it to a known theorem and then extend it to the limit of infinite number of degrees of freedom. In quantum mechanics, we prove GGE for maximal Hamiltonians—a class of models stemming from a rigorous notion of quantum integrability understood as the existence of conserved charges with prescribed dependence on a system parameter, e.g. Hubbard U, anisotropy in the XXZ model etc. In analogy with classical integrability, the defining property of these models is that they have the maximum number of independent integrals. We contrast their dynamics induced by quenching the parameter to that of random matrix Hamiltonians.

The past decade has witnessed an unprecedented experimental access to global coherent dynamics of many-body interacting systems [1–7]. As a result, a new area that could be called "far from equilibrium many-body Hamiltonian dynamics", "coherent many-body dynamics" or "quantum quenches" has emerged. A major part of research in this area has focused on testing the GGE [8,9] in various integrable models. GGE refers to a density matrix or, in the case of classical mechanics, a

E-mail address: eyuzbash@physics.rutgers.edu.

http://dx.doi.org/10.1016/j.aop.2016.02.002
0003-4916/

phase space distribution function

$$\rho = Z^{-1} e^{-\sum_k \beta_k H_k}, \tag{1}$$

where H_k are a (complete in some sense) set of integrals of motion for system Hamiltonian H and Z is a normalization constant. Suppose the system evolves with H starting from a non-stationary state. The statement of GGE is that the infinite time average of an observable O coincides with its ensemble average with the density matrix ρ [10].

Most authors test GGE in quantum models without clarifying their notion of quantum integrability. The latter however is a tricky concept with no generally accepted definition, making the quantum GGE conjecture essentially unfalsifiable. The notion of classical integrability on the other hand is unambiguous [11]. For this and other reasons, it makes sense to first understand the status of GGE in classical mechanics. We will see that the microcanonical version of GGE – Generalized Microcanonical Ensemble – is exact for a general classical integrable Hamiltonian. In a parallel line of inquiry, we will prove GGE for a class of models that emerge from a recently proposed complete notion of quantum integrability.

Generalized Microcanonical Ensemble (GME) in classical mechanics is the following phase space distribution:

$$\rho(\boldsymbol{p}, \boldsymbol{q}) = L^{-1} \prod_{k=1}^{n} \delta\left(H_k(\boldsymbol{p}, \boldsymbol{q}) - h_k\right), \tag{2}$$

where $\boldsymbol{q} = (q_1, \ldots, q_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ are the generalized coordinates and momenta and L is a normalization constant. Suppose the system evolves with an integrable Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$ starting from a point $(\boldsymbol{p}_0, \boldsymbol{q}_0)$. Let $h_k = H_k(\boldsymbol{p}_0, \boldsymbol{q}_0)$ be the values of its integrals of motion for this initial condition. The statement of GME is that the time average of any dynamical variable $O(\boldsymbol{p}, \boldsymbol{q})$ is equal to its phase space average with distribution (2),

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} O(t)\, dt = \int O(\boldsymbol{p}, \boldsymbol{q}) \rho(\boldsymbol{p}, \boldsymbol{q}) d\boldsymbol{p} d\boldsymbol{q}, \tag{3}$$

where $O(t) = O(\boldsymbol{p}(t), \boldsymbol{q}(t))$. Eq. (3) also holds for integrable classical spin Hamiltonians $H(\{\vec{s}_k\})$, in which case $p_k = \cos\theta_k$ and $q_k = \phi_k$, where θ_k and ϕ_k are the polar and azimuthal angles defining the spin direction. Eq. (3) is valid for any number of degrees of freedom n, so one can take the limit $n \to \infty$ on both sides. Moreover, we will argue that the limits $n \to \infty$ and $\tau \to \infty$ commute (a tremendous simplification) as long as the frequency spectrum of $O(t)$ is free from a certain anomaly near the zero frequency.

As a first step towards a similarly unambiguous statement in quantum mechanics, we also analyze GGE in the framework of a rigorous formulation of quantum integrability [12]. Simplest models that arise in this approach are type-1 or maximal Hamiltonians—general N linearly independent commuting $N \times N$ Hermitian matrices of the form $H(x) = T + xV$, where x is a real parameter. Type-1 matrices represent blocks of various exactly solvable many-body models (such as 1D Hubbard and Gaudin magnets) for certain sets of quantum numbers (total spin projection etc.) [12–15] and also describe e.g. a short range impurity in a metallic grain [16], see below for more detail. We prove GGE is exact for any N and explicitly determine β_k in Eq. (1). The GGE density matrix for quenches of the parameter x turns out to be non-thermal for type-1 Hamiltonians. In contrast, if we choose T and V randomly, the post-quench asymptotic state is thermal in $N \to \infty$ limit. This emphasizes the importance of a well-defined notion of integrability as naively one could claim N integrals of motion (e.g. projectors onto the eigenstates) in the random matrix example too. We also relate the non-thermal behavior to localization.

A characteristic feature of type-1 and classical integrable systems is that in both cases the number of independent integrals is the maximum allowed by the definition. Their dynamics are constrained by the integrals apart from linear in time phases (angles) that cancel out upon time-averaging or dephase in the thermodynamic limit. As the result, the integrals of motion fully determine infinite time averages. The situation when the number of conservation laws is appreciably less than the