Annals of Physics 366 (2016) 22–31

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

A new entropy based on a group-theoretical structure

Evaldo M.F. Curado [a,b,*], Piergiulio Tempesta [c,d], Constantino Tsallis [a,b,e]

[a] Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290–180 Rio de Janeiro – RJ, Brazil

[b] National Institute of Science Technology for Complex Systems, Rua Xavier Sigaud 150, 22290–180 Rio de Janeiro – RJ, Brazil

[c] Departamento de Física Teórica II (Métodos Matemáticos de la Física), Facultad de Físicas, Universidad Complutense, 28040 – Madrid, Spain

[d] Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain

[e] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

ARTICLE INFO

Article history:
Received 10 July 2015
Accepted 25 December 2015
Available online 4 January 2016

Keywords:
Generalized entropies
Group theory
Statistical mechanics

ABSTRACT

A multi-parametric version of the nonadditive entropy S_q is introduced. This new entropic form, denoted by $S_{a,b,r}$, possesses many interesting statistical properties, and it reduces to the entropy S_q for $b = 0, a = r := 1 - q$ (hence Boltzmann–Gibbs entropy S_{BG} for $b = 0, a = r \to 0$). The construction of the entropy $S_{a,b,r}$ is based on a general group-theoretical approach recently proposed by one of us, Tempesta (2016). Indeed, essentially all the properties of this new entropy are obtained as a consequence of the existence of a rational group law, which expresses the structure of $S_{a,b,r}$ with respect to the composition of statistically independent subsystems. Depending on the choice of the parameters, the entropy $S_{a,b,r}$ can be used to cover a wide range of physical situations, in which the measure of the accessible phase space increases say exponentially with the number of particles N of the system, or even stabilizes, by increasing N, to a limiting value.

* Corresponding author at: Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290–180 Rio de Janeiro – RJ, Brazil.

E-mail addresses: evaldo@cbpf.br (E. M.F. Curado), p.tempesta@fis.ucm.es, piergiulio.tempesta@icmat.es (P. Tempesta), tsallis@cbpf.br (C. Tsallis).

http://dx.doi.org/10.1016/j.aop.2015.12.008

This paves the way to the use of this entropy in contexts where the size of the phase space does not increase as fast as the number of its constituting particles (or subsystems) increases.

1. Introduction

In the last decades, the non-extensive scenario, originally proposed in [1], has been largely investigated as a new thermodynamic framework allowing to generalize the standard Boltzmann–Gibbs approach to new physical contexts, where ergodicity hypothesis is violated [2]. At the same time, the ubiquity of the notion of entropy in social sciences paved the way to fruitful extensions of the standard information theory of Shannon and Khinchin [3–5] towards new classical and quantum formulations.

The search for new entropic forms has been very active in the last decades. Many different entropies have been proposed, generalizing the Boltzmann–Gibbs entropy from different perspectives (see, e.g., [6–17]).

In particular, a group-theoretical approach to the notion of entropy has been advocated in [14]. It is based on the observation that a thermodynamically admissible entropy should satisfy not only the first three Khinchin axioms (continuity, concavity, expansibility), but also a general *composability property*. It amounts to require that, given an entropic functional S, its values on a system defined by the union of two statistically independent subsystems A and B should depend (in addition to a possible set of fixed indices; for instance the index q for S_q) on the entropies of the two subsystems only. This requirement is motivated by the fact that entropy makes sense for macroscopic objects.

Composability can be imposed on full generality (and we shall talk about *strict composability* or composability tout court) or at least on subsystems characterized by the uniform distribution (composability in a *weak sense*). This last property applies, for instance, when considering isolated physical systems at the equilibrium (microcanonical ensemble), or in contact with a thermostat at very high temperature (canonical ensemble).[1] In full generality, it amounts to say that there exists a smooth function of two real variables $\Phi(x, y)$ such that (C1)

$$S(A \cup B) = \Phi(S(A), S(B); \{\eta\}) \tag{1}$$

where $\{\eta\}$ is a possible set of real continuous parameters, and $A \subset X$ and $B \subset X$ are two statistically independent subsystems of a given system X, with the further properties

(C2) Symmetry:

$$\Phi(x, y) = \Phi(y, x). \tag{2}$$

(C3) Associativity:

$$\Phi(x, \Phi(y, z)) = \Phi(\Phi(x, y), z). \tag{3}$$

(C4) Null-composability:

$$\Phi(x, 0) = x. \tag{4}$$

Note that the associativity property is crucial for the applicability of the zeroth law of thermodynamics. The Boltzmann–Gibbs, the S_q, the Rényi entropies and the non-trace form Z entropies [15] are known to be strictly composable. Instead, the weak composability property is shared by infinitely

[1] To be more precise, let us illustrate the BG case. The formula $S_{BG} = \ln W$ (assuming W to be finite) applies to both microcanonical and $T \to \infty$ canonical cases, but, in the former, W refers to the total number of states within a thin slice of phase space corresponding to a given total energy, whereas, in the latter, W refers to the total number of states within the entire phase space.