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# Entanglement entropy converges to classical entropy around periodic orbits

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ABSTRACT

We consider oscillators evolving subject to a periodic driving force that dynamically entangles them, and argue that this gives the linearized evolution around periodic orbits in a general chaotic Hamiltonian dynamical system. We show that the entanglement entropy, after tracing over half of the oscillators, generically asymptotes to linear growth at a rate given by the sum of the positive Lyapunov exponents of the system. These exponents give a classical entropy growth rate, in the sense of Kolmogorov, Sinai and Pesin. We also calculate the dependence of this entropy on linear mixtures of the oscillator Hilbert-space factors, to investigate the dependence of the entanglement entropy on the choice of coarse graining. We find that for almost all choices the asymptotic growth rate is the same.



## 1. Introduction

The results of [1] state that, roughly speaking, if we have a quantum system with a finite-dimensional Hilbert space of states that we factorize into a product of Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B, \tag{1}$$

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then the typical pure state in $\mathcal{H}$ has very close to the maximal amount of entanglement allowed between $\mathcal{H}_A$ and $\mathcal{H}_B$, and this is in turn maximized if $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$. We will call such factorization of $\mathcal{H}$ into "observable" and "non-observable" physics a coarse-graining of the system. This suggests that if we evolve a system randomly from an initial configuration with zero entanglement entropy, then it will eventually forget essentially all of the information of the initial state if we only measure observables sensitive to $\mathcal{H}_A$. There are many studies of this kind of process in specific systems. The rate at which the entanglement grows towards saturation depends, in general, on the details of these systems. However, some general bounds exist [2] and linear growth in time appears in many studies of decoherence and quantum chaos [3–16].

When trying to apply these results in the context of black hole physics, we are usually confronted with two basic problems. First of all, the Hilbert space $\mathcal{H}$ is big. In the gauge/gravity duality [17] the dynamics takes place in an infinite-dimensional Hilbert space: it is the Hilbert space of a relativistic quantum field theory on the conformal boundary.

A very naive application of the results of [1] would suggest that typical states have infinite entropy when splitting $\mathcal{H}$ in two pieces of the same size, since both are infinite dimensional. However, the notion of splitting along a random factorization has no meaning, because once we have factored into infinite dimensional pieces, we can factorize the pieces again: there is no natural notion of splitting in half. Thus, the question of the entanglement entropy for a typical state is ill-defined without additional structure on the Hilbert space.

An example of such a structure is two operator algebras, one for $\mathcal{H}_A$ the other one for $\mathcal{H}_B$. It is natural to do the splitting with respect to a choice of algebras with reasonable properties determined by features of the dynamics. Once that splitting is done, instead of computing the entanglement entropy of the typical state, we can compute the rate of growth of the entanglement entropy as governed by the dynamics and ask how this growth is affected by our choices of coarse-graining and dynamics. It is here that we need a model dynamics that is both tractable and generic. We have in mind two simple harmonic oscillators with two ladder operator algebras, and we will show in what sense a system like this can be considered generic.

The second problem we generically find is that there is no obvious canonical splitting into two factors $\mathcal{H}_A$ and $\mathcal{H}_B$, so one might expect that entanglement entropy based on some such splitting might depend substantially on the coarse-graining. If we also define the scrambling rate as the slope of the entropy growth, one might worry that there is no objective way to quantify it. This would make it very hard to understand in what sense black holes are fast scramblers [18,19], when we think of the evolution in terms of a dual quantum field theory. In this sense, it is natural to ask if there is a universal result where the details of the factorization do not matter too much. Our main motivation is to eventually formulate the fast scrambling conjecture on a rigorous footing, but to do so, we need to be able to apply the methods that could characterize scrambling to fairly generic dynamical systems to which we associate an infinite-dimensional Hilbert space.

The purpose of this paper is to analyze the scrambling rate, i.e., the entropy growth rate, in a toy model of chaotic dynamics that iterates a relatively simple unitary evolution operator on an infinite-dimensional Hilbert space $\mathcal{H}$. The Hilbert space will be further decomposed into a product of two infinite-dimensional Hilbert spaces. This is done for a closed system, and we will study the dependence on the choice of coarse-graining and initial state. The idea is to study the quantum counterparts of closed Hamiltonian chaotic dynamical systems with finitely many degrees of freedom, similarly to previous studies of closed systems in the quantum chaos and decoherence literature [6,9,13].

So long as these classical systems have bounded trajectories (for example if the regions with bounded energy have finite volume), then they are expected to have a dense set of periodic trajectories. We assume that these periodic trajectories encode all the important information of the dynamical system, and that any classical initial condition is sufficiently close to such a periodic orbit, in line with periodic-orbit theory in quantum chaos [20–22]. The evolution of the classical system for such an initial condition can then be understood, for some time, by the linearized perturbations around the corresponding periodic orbit. We can ask how these perturbations grow in time and estimate the Lyapunov exponents of the full system from such an analysis. Given such Lyapunov