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# Stable compact motions of a particle driven by a central force in six-dimensional spacetime

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ABSTRACT

We give a counterexample to the well-known Ehrenfest's assertion that the existence of stable electromagnetic bound systems is impossible in spaces of more than three dimensions. If we require that the Maxwellian laws of electromagnetism be preserved for any even spacetime dimension, and that the dynamics as a whole be consistent, then the laws of mechanics must be amended by the addition of terms with higher derivatives. We consider a nonrelativistic particle with an acceleration-dependent Lagrangian which moves in an attractive $1/r^3$ potential in five-dimensional space. There are compactly supported motions whose projections on the SO(5)-reduced Hamiltonian system are Poisson equilibrium points. The nonlinearly stable equilibria correspond to physically stable motions over the direct product of two three-spheres in configuration space. The Energy-Casimir method turns out to be not appropriate for checking the stability. The studied system is shown to be stable through an analysis of numerical solutions to the equations of motion for small perturbations on the reduced phase space. This implies that falling to the center is prevented.



## 1. Introduction

It is widely believed that a charged particle driven by an attractive electrostatic central force in $D$-dimensional space $\mathbb{R}_D$ is unable to execute compactly supported stable motions if $D \geq 4$; all motions are compactly supported if $D \leq 2$; and both compactly and noncompactly supported motions

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of charged particles are peculiar to $\mathbb{R}_3$. This belief goes back to Ehrenfest's analysis of the question: "By which singular characteristics do geometries and physics in $\mathbb{R}_3$ distinguish themselves from those in the other $\mathbb{R}_D$'s?" raised in his 1917 and 1920 papers [1,2]. To answer this question Ehrenfest assumed that the laws of mechanics and electrodynamics in an imaginary world with $D$ spatial coordinates are essentially the same as those in $\mathbb{R}_3$, or, more exactly, the laws governing a closed system of $N$ charged point particles are encoded into the conventional action

$$S = -\sum_{I=1}^{N} \int d\tau_I \left[ m_0^I \sqrt{\dot{x}_I \cdot \dot{x}_I} + e_I \dot{x}_I^\mu A_\mu(x_I) \right] - \frac{1}{4\Omega_{n-1}} \int d^{D+1}x F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where $\Omega_{D-1}$ is the area of the unit $(D-1)$-sphere. In fact, one need only look into the behavior of two particles with charges $Ze$ and $-e$. Ehrenfest supposed that, for any $D$, the two-particle problem can be reduced to a single-particle Kepler problem which describes the behavior of a particle of reduced mass $\bar{m}$ and charge $-e$ in a spherically symmetric attractive field generated by a static charge $Ze$ placed at the origin. A distinctive feature of this problem is that every trajectory is planar. The qualitative analysis of the Kepler problem is greatly facilitated with the aid of the effective potential (see, e.g., [3])

$$\mathcal{U}(r) = \sqrt{\bar{m}^2 + \frac{\mathrm{L}^2}{r^2}} + U(r). \tag{2}$$

Here, $\mathrm{L}^2 = \sum_{i<j} L_{ij}^2$ is the square of angular momentum, and $U(r) = e\phi(\mathbf{x})$ the potential energy of interaction between the particles. The time component of the electromagnetic vector potential $\phi(\mathbf{x})$ obeys the $D$-dimensional Poisson equation

$$\nabla^2 \phi(\mathbf{x}) = -\Omega_{D-1}\, Ze\, \delta^{(D)}(\mathbf{x}). \tag{3}$$

The solution to Eq. (3) is

$$\phi(\mathbf{x}) = -Ze \begin{cases} \mathrm{sgn}(2-D)\, |\mathbf{x}|^{2-D} & D \neq 2, \\ \log |\mathbf{x}| & D = 2. \end{cases} \tag{4}$$

For $D > 3$, the potential energy $e\phi(\mathbf{x})$ is more singular than the centrifugal term $|\mathrm{L}|/r$, and falling to the center (or, alternatively, going to infinity) is unavoidable, while, for $D = 3$ and $Ze^2 \le |\mathrm{L}|$, $e\phi(\mathbf{x})$ is less singular than $|\mathrm{L}|/r$, which prevents falling to the center, so that stable orbits are possible. This has led Ehrenfest to conclude that $D = 3$ establishes a line of demarcation between worlds where stable bound systems such as a hydrogen atom cannot exist from those where such systems are possible.[1]

Clearly this analysis is oversimplified because the interaction between the particles is represented by the electrostatic potential (4), and the retardation effect is neglected. If radiation of the planetary electron were taken into account, then this electron would fall to the nucleus even in $\mathbb{R}_3$. To remedy the situation, Ehrenfest invoked the Bohr quantization which evidences that the hydrogen atom is stable in real space due to its quantum nature. Ehrenfest found that the Bohr model of the atom in $\mathbb{R}_D$ with $D \ge 5$ exhibits the spectrum of positive discrete energy levels, $E_n \sim \frac{D-2}{D-4}\, n^{2(D-2)/(D-4)}$. This suggests that the system tends to go to lower and lower energy levels which corresponds to ever increasing radii of the associated orbital motion, to yield finally the ionization of the atom.

A more rigorous analysis of this problem confirming the Ehrenfest's general conclusion was proposed by Gurevich and Mostepanenko [4]. They studied a quantum system of two particles whose interaction is given by (4), using the conventional Schrödinger equation. They showed that, for $D \ge 4$, the discrete energy spectrum extends to $-\infty$. It has long been known [5] that such systems tends

[1] We use the term "stability" rather loosely because compact motions in the central force problem are unstable in the sense of Lyapunov. For example, a small perturbation of radius of a Kepler orbit eventually results in that the perturbed and initial orbits become 180° out of phase, which means that this motion is unstable in the Lyapunov sense. Intuitively, an orbiting is stable if a small perturbation of the solution to the equations of motion leaves it in a compact region of the phase space, with the understanding that the fall to the center is prevented. In the Kepler problem we actually deal with just this stability rather than Lyapunov stability. We will call such compact motions physically stable.