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Entanglement entropy and variational methods: Interacting scalar fields

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ABSTRACT

We develop a variational approximation to the entanglement entropy for scalar ϕ^4 theory in $1+1$, $2+1$, and $3+1$ dimensions, and then examine the entanglement entropy as a function of the coupling. We find that in $1+1$ and $2+1$ dimensions, the entanglement entropy of ϕ^4 theory as a function of coupling is monotonically decreasing and convex. While ϕ^4 theory with positive bare coupling in $3+1$ dimensions is thought to lead to a trivial free theory, we analyze a version of ϕ^4 with infinitesimal negative bare coupling, an asymptotically free theory known as *precarious* ϕ^4 theory, and explore the monotonicity and convexity of its entanglement entropy as a function of coupling. Within the variational approximation, the stability of precarious ϕ^4 theory is related to the sign of the first and second derivatives of the entanglement entropy with respect to the coupling.

1. Introduction

Entanglement is a pervasive phenomenon in quantum physics, and has been studied extensively in finite-dimensional systems using the tools of quantum information theory. More elusive is the entanglement structure of infinite-dimensional systems, namely quantum field theories. The last twenty years has seen the development of a large literature on the entanglement entropy of QFT's and

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has shed light on the entanglement between spatial degrees of freedom [1–4]. Currently, there are only a few classes of field theories for which entanglement entropy can be computed exactly. These include certain exactly solvable QFT's, free field theories, and conformal field theories. CFT's with holographic duals are particularly interesting because their entanglement entropies can be computed using the Ryu–Takayanagi formula [5].

Despite this vast body of work, the entanglement entropy of QFT's remains rather mysterious. Entanglement entropies for QFT's typically contain a combination of divergent and finite terms which depend on the parameters of the QFT. Often it can be difficult to gain insight into the entanglement structure of a QFT by analyzing the form of its entanglement entropy. More insightful is how the entanglement entropy *changes* as a QFT undergoes dynamics such as a quench, or as parameters of the QFT are tuned.

Conspicuously missing from the list of QFT's with exactly computable entanglement entropies are those interacting field theories which comprise the Standard Model. Even conventional QFT's such as ϕ^4 theory and Yukawa theory have not had much presence in the entanglement entropy literature. While it may not be possible to study the exact entanglement entropy of these theories, it is possible to study perturbative or variational approximations.

In this paper we embark on a program to analyze the entanglement entropies of more conventional interacting QFT's, and begin with interacting scalar field theories and ϕ^4 theory in particular. Our plan of attack is to use a variational principle to determine a non-perturbative approximation to the ground state of ϕ^4 theory for arbitrary coupling, and then compute the entanglement entropy of the approximation. Our variational ansatz will be a Gaussian wave functional, for which we can compute the entanglement entropy exactly. Since the ground state of a massive free scalar field theory is exactly a Gaussian wave functional, our approximation will be accurate in a neighborhood of zero coupling. We will also analyze the variational approximation at larger values of the coupling, for which the variational method is well-defined. The validity and accuracy of the Gaussian variational approximation compares favorably to one-loop computations and large N approximations for theories where such comparisons make sense. Our consideration of the approximation at larger values of the coupling will augment our understanding of the small coupling case, and provide evidence for features of entanglement entropy that may hold for all values of the coupling.

Once we have computed the variational approximation to the entanglement entropy in ϕ^4 theory, we will analyze how it changes as we tune the coupling. We will find that the derivatives of entanglement entropy with respect to coupling are often finite and independent of regularization, and represent meaningful physical quantities. By studying the dependence of entanglement entropy on the coupling, we find several surprising features which suggest new insights into the nature of entanglement in QFT's.

The structure of the paper is as follows: First we discuss the variational methods used to approximate the ground state and other more general states of ϕ^4 theory. Then we explain how to compute the entanglement entropy of the variational ansatz using the replica trick. Finally, we analyze the dependence of the entanglement entropy on the coupling in various dimensions for ϕ^4 theory within the variational approximation.

2. Variational methods and the Gaussian effective potential

In this section we introduce and review the essential aspects of the variational methods we will use for the approximate calculations of entanglement entropy in interacting quantum field theories. The framework of variational methods that employ a Gaussian trial wave functional in the computation of the effective action and effective potential (in particular the 2PI effective action; see Cornwall, Jackiw, and Tomboulis [6]) is referred to as the *Gaussian Effective Potential*, or GEP. The GEP for ϕ^4 theory, as well as for other quantum field theories, has been studied extensively by Stevenson and his collaborators [7–10] beginning in the 1980s. That work built on earlier work of Barnes and Ghandour [11] and others in the 1970s, which in turn extended and clarified the pioneering work of Schiff [12] in the early 1960s. We will trace the development of the GEP and the closely related gap equation along a slightly different route, using a special case of a time-dependent variational principle based on work of Kerman and Koonin [13], and applied to quantum field theory by Jackiw, Kerman,